ELIZABETHTOWN COLLEGE
Putnam Preparation Series
B. Doytchinov, 2007

# Introduction to the Putnam Contest

## 1. ABOUT THE CONTEST

The William Lowell Putnam Mathematical Competition, probably the most famous, North American intercollegiate math contest, is held annually on the first Saturday in December. This year's contest (the 68th) has been now officially announced, and will take place on Saturday, December 1, 2007.

The contest is open to full-time undergraduate students who have not yet received a degree.

Since the competition emphasizes ingenuity rather than knowledge, freshmen are not much at a disadvantage compared to seniors. Although it is a math contest, it is not limited to math students. Some recent winners came from nearby disciplines, including physics, computer science, and engineering.

The competition consists of a morning session (3 hours, 6 questions A1-A6) and an afternoon session (3 hours, 6 more questions B1-B6). Each question is worth 10 points, so the maximum total is 120 points. The median score is often 0, 1, or 2 out of 120; in 2003 the median was 1, in 2004 it was 0, in 2005 again 1, and in 2006 again 0. In fact, in 2006 close to 2/3 of all participants got a score of 0.

The problems rarely require heavy computations or a lot of knowledge, but they are difficult because they are "non-standard". They are "proof questions": an answer is not enough, it must be supported by a rigorous proof (not just "beyond reasonable doubt"). For sample questions, see the attached 1998 competition. The median on that competition was 10 (the largest for the last 10 years).

Each problem is graded out of 10 points, however the score can only be 0, 1, 2, 8, 9, or 10 points (i.e. there are no scores in the "middle range"). A score of 2 is for significant progress, and a score of 8 means an almost perfect solution. This suggests the following strategy: it is better to pick just one or two problems and write them really well, then to try to get a lot of partial credit on many problems (because you won't). Contestants who complete just one problem correctly, usually will rank somewhere in the top 20% to 50% of all who take the exam, and those working out two problems correctly, often rank in the top 10% to 20% (inter)nationally.

## 2. PREPARATION

There are various resources you can avail yourself of:

- people in the department (faculty and other students) who will be happy to talk with you;
- the more recent Putnam exams (and solutions), easily available in the library or on the web (see the link below);
- if there is interest, we can organize more preparation meetings, for updates and times, see `http://users.etown.edu/d/doytchinovb/putnam/sessions.html`

**Books:**

- *Problem Solving through Problems*, by Loren Larson and Paul Halmos (ISBN 0-38796171-2);
- *The William Lowell Putnam Mathematical Competition problems and solutions : 1965-1984*, by Gerald L. Alexanderson, Leonard F. Klosinski, Loren C. Larson (ISBN 0-88385-441-4);
- *The William Lowell Putnam Mathematical Competition 1985-2000: Problems Solutions, and Commentary*, by Kiran S. Kedlaya, Bjorn Poonen, and Ravi Vakil (ISBN 0-88385-087-X);
- *Which Way Did the Bicycle Go?*, by Joseph D. E. Konhauser, Dan Velleman, and Stan Wagon (ISBN 0-88385-325-6).
- *The Green Book of Mathematical Problems*, by K. Hardy and K. Williams, (ISBN 0-486-69573-5);
- *The Red Book of Mathematical Problems*, by K. Hardy and K. Williams, (ISBN 0-486-69415-1).

**Online resourses:**
For links, see my web page:
`http://users.etown.edu/d/doytchinovb/putnam/resource.html`

# The 59th William Lowell Putnam Mathematical Competition

**Saturday, December 5, 1998**

**A1.** A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

**A2.** Let $s$ be any arc of the unit circle lying entirely in the first quadrant. Let $A$ be the area of the region lying below $s$ and above the $x$-axis and let $B$ be the area of the region lying to the right of the $y$-axis and to the left of $s$. Prove that $A + B$ depends only on the arc length, and not on the position, of $s$.

**A3.** Let $f$ be a real function on the real line with continuous third derivative. Prove that there exists a point $a$ such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$

**A4.** Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number $A_n$ is defined by concatenating the decimal expansions of $A_{n-1}$ and $A_{n-2}$ from left to right. For example $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all $n$ such that 11 divides $A_n$.

**A5.** Let $\mathcal{F}$ be a finite collection of open discs in $\mathbb{R}^2$ whose union contains a set $E \subseteq \mathbb{R}^2$. Show that there is a pairwise disjoint subcollection $D_1, \ldots, D_n$ in $\mathcal{F}$ such that

$$E \subseteq \cup_{j=1}^{n} 3D_j.$$

Here, if $D$ is the disc of radius $r$ and center $P$, then $3D$ is the disc of radius $3r$ and center $P$.

**A6.** Let $A, B, C$ denote distinct points with integer coordinates in $\mathbb{R}^2$. Prove that if

$$(|AB| + |BC|)^2 < 8 \cdot [ABC] + 1$$

then $A, B, C$ are three vertices of a square. Here $|XY|$ is the length of segment $XY$ and $[ABC]$ is the area of triangle $ABC$.

**B1.** Find the minimum value of

$$\frac{(x+1/x)^6-(x^6+1/x^6)-2}{(x+1/x)^3+(x^3+1/x^3)}$$

for $x>0$.

**B2.** Given a point $(a,b)$ with $0<b<a$, determine the minimum perimeter of a triangle with one vertex at $(a,b)$, one on the $x$-axis, and one on the line $y=x$. You may assume that a triangle of minimum perimeter exists.

**B3.** Let $H$ be the unit hemisphere $\{(x,y,z): x^2+y^2+z^2=1, z\geq 0\}$, $C$ the unit circle $\{(x,y,0): x^2+y^2=1\}$, and $P$ the regular pentagon inscribed in $C$. Determine the surface area of that portion of $H$ lying over the planar region inside $P$, and write your answer in the form $A\sin\alpha+B\cos\beta$, where $A,B,\alpha,\beta$ are real numbers.

**B4.** Find necessary and sufficient conditions on positive integers $m$ and $n$ so that

$$\sum_{i=0}^{mn-1}(-1)^{\lfloor i/m\rfloor+\lfloor i/n\rfloor}=0.$$

**B5.** Let $N$ be the positive integer with 1998 decimal digits, all of them 1; that is,

$$N=1111\cdots 11.$$

Find the thousandth digit after the decimal point of $\sqrt{N}$.

**B6.** Prove that, for any integers $a,b,c$, there exists a positive integer $n$ such that $\sqrt{n^3+an^2+bn+c}$ is not an integer.