

THEME 1

1. TOPICS

Today's practice problems are not of any particular type. Most of these were chosen from the "easier" problems of actual Putnam exams. In fact, some of them are so straightforward and run-of-the-mill that it is hard to believe they are "Putnam" problems.

2. ADVICE OF THE PROS

On Stanford's Putnam preparation site, Ravi Vakil gives the following advice on how to approach the Putnam preparation problems:

Work in groups. Try small cases. Do examples. Look for patterns. Draw pictures. Use lots of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backwards. Argue by contradiction. Consider extreme cases. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask. If the problem has a 2007 in it, what happens if you replace 2007 by 1, or 2, or 3? What is important about 2007? Is it that it is not prime, or that it is odd, divisible by 9, etc.?

3. PRACTICE PROBLEMS

1. **(1998A1)** A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?
2. **(2005B1)** Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a . (Note: $\lfloor \nu \rfloor$ is the greatest integer less than or equal to ν .)
3. **(2006B1)** Show that the curve $x^3 + 3xy + y^3 = 1$ contains only one set of three distinct points, A , B , and C , which are vertices of an equilateral triangle, and find its area.

4. **(2005B5)** For each continuous function $f : [0, 1] \rightarrow \mathbb{R}$, let $I(f) = \int_0^1 x^2 f(x) dx$ and $J(f) = \int_0^1 x (f(x))^2 dx$. Find the maximum value of $I(f) - J(f)$ over all such functions f .

5. **(1977B1)** Evaluate the infinite product

$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}.$$

6. **(1984B1)** Let n be a positive integer, and define

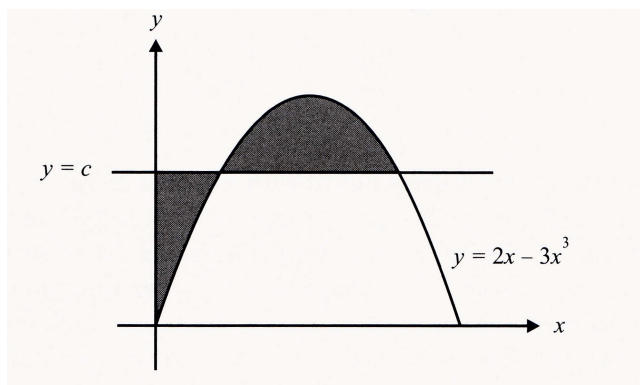
$$f(n) = 1! + 2! + \cdots + n!.$$

Find polynomials P and Q such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n)$$

for all $n \geq 1$.

7. **(1993A1)** The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find c so that the areas of the two shaded regions are equal.



8. **(1995A1)** Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S , then so is ab). Let T and U be disjoint subsets of S whose union is S . Given that the product of any *three* (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U , show that at least one of the two subsets T, U is closed under multiplication.

9. **(2004A1)** Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N , but by the end of the season, $S(N)$ was more than 80% of N . Was there necessarily a moment in between when $S(N)$ was exactly 80% of N ?
10. **(1984A1)** Let A be a solid $a \times b \times c$ rectangular brick in three dimensions, where $a, b, c > 0$. Let B be the set of all points that are a distance at most one from some point of A (in particular, B contains A). Express the volume of B as a polynomial of a , b , and c .
11. **(1991A2)** Let \mathbf{A} and \mathbf{B} be different $n \times n$ matrices with real entries. If $\mathbf{A}^3 = \mathbf{B}^3$ and $\mathbf{A}^2\mathbf{B} = \mathbf{B}^2\mathbf{A}$, can $\mathbf{A}^2 + \mathbf{B}^2$ be invertible?
12. **(Bcl163)** Believe it or not, there are numbers a and b , with $a < b$, such that the expression

$$\sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}}$$

is constant for $a \leq x \leq b$. Find a and b , and the constant value of the expression.

13. **(1989B1)** A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $\frac{a\sqrt{b} + c}{d}$, where a , b , c , d are integers.
14. **(1988A1)** Let R be the region consisting of the points (x, y) of the cartesian plane satisfying both $|x| - |y| \leq 1$ and $|y| \leq 1$. Sketch the region R and find its area.
15. **(1991A1)** A 2×3 rectangle has vertices at $(0, 0)$, $(2, 0)$, $(0, 3)$, and $(2, 3)$. It rotates 90° clockwise about the point $(2, 0)$. It then rotates 90° clockwise about the point $(5, 0)$, then 90° clockwise about the point $(7, 0)$, and finally, 90° clockwise about the point $(10, 0)$. (The side originally on the x -axis is now back on the x -axis.) Find the area of the region above the x -axis and below the curve traced out by the point whose initial position is $(1, 1)$.

16. **(1986A1)** Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.
17. **(1986B1)** Inscribe a rectangle of base b and height h in a circle of radius one, and inscribe an isosceles triangle in the region of the circle cut off by one base of the rectangle (with that side as the base of the triangle). For what value of h do the rectangle and triangle have the same area?
18. **(2001B2)** Find all pairs of real numbers (x, y) satisfying the system of equations

$$\begin{aligned}\frac{1}{x} + \frac{1}{2y} &= (x^2 + 3y^2)(3x^2 + y^2) \\ \frac{1}{x} - \frac{1}{2y} &= 2(y^4 - x^4).\end{aligned}$$

19. **(1999A1)** Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that for all x ,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \leq x \leq 0 \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

20. **(1998A2)** Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x -axis and let B be the area of the region lying to the right of the y -axis and to the left of s . Prove that $A + B$ depends only on the arc length, and not on the position, of s .
21. **(1994A1)** Suppose that a sequence a_1, a_2, a_3, \dots satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that $\sum_{n=1}^{\infty} a_n$ diverges.
22. **(1994B1)** Find all positive integers that are within 250 of exactly 15 perfect squares.
23. **(ARML1991T5)** Compute the number of real values of x that satisfy the equation

$$\left| |x^2 - 1| - 1 \right| = 2^x.$$

24. **(ARML1991T6)** One angle of a triangle is twice another, and the sides opposite these angles have lengths 15 and 9. Compute the length of the third side of the triangle.
25. **(UDN090)** Find the shortest distance between the circle $x^2 + y^2 = 5$ and the line $y = -2x + 10$.
26. **(UDN119)** Show that the sequence $a_n = n \cos n$ is not bounded. Is the sequence $b_n = n \sin n$ bounded?
27. **(UDN121)** The sequence $a_1, a_2, \dots, a_n, \dots$ consists entirely of the numbers 0, 1, and 2, and is not periodic. Define two more sequences as follows:

$$b_n = \begin{cases} 0 & \text{if } a_n = 0, \\ 1 & \text{otherwise} \end{cases} \quad c_n = \begin{cases} 1 & \text{if } a_n = 2, \\ 0 & \text{otherwise} \end{cases}$$

Show that at least one of the sequences b_n and c_n is not periodic.