ELIZABETHTOWN COLLEGE
Putnam Preparation Series
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THEME 2

1. TOPICS

Today I will talk about three techniques that prove useful.

The pigeonhole principle is a favorite among authors of math contest problems. It requires no advanced math, yet sometimes it has far reaching consequences.

The principle of mathematical induction is a useful "brute force" machine for proving statements of the form "For all positive integers n ..." It is rare to have a whole problem on the contest that can be solved by induction alone. Usually one must do some preliminary work to reduce the problem (or part of it) to an induction argument. For this reason, some of today's practice problems dealing with induction are simpler than a regular Putnam problem.

The principle of the minimal counterexample supersedes the principle of induction, and works in more complicated situations.

2. PRACTICE PROBLEMS

The first problems are for practice with the pigeonhole principle. The pigeonhole principle can be described, in short, by the following observation: if we distribute $n + 1$ rabbits into n baskets, then there must be at least one basket with two or more rabbits. More generally, if we put $nk + 1$ rabbits into n baskets, then at least one basket has at least $k+1$ rabbits. Sometimes, some ingenuity is required in order to realize that the pigeonhole principle can be applied to the situation at hand.

1. Prove that there are two people in the U.S. right now with the same amount of hair on their heads (not including bald people!).

2. **(2002A2)** Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

3. Given any $n + 1$ integers, show that there always can be chosen two of them so that their difference is divisible by n.

4. Given a finite sequence of n integers, $a_1, a_2, \ldots, a_n$, show that we can always pick several consecutive terms (could be just one) so that their sum is divisible by n.

5. Let A be any set of 20 distinct integers chosen from the arithmetic progression $1,\ 4,\ 7,\ \ldots,\ 100$. Prove that there must be two distinct integers in A whose sum is 104.

6. **(Eng4E2)** A chessmaster has 77 days to prepare for a tournament. He wants to play at least one game per day, but not more than 132 games altogether. Prove that there is a sequence of successive days during which he plays exactly 21 games.

7. **(Eng4P15)** Twenty distinct positive integers are all strictly less than 70. Prove that among their pairwise differences there are four equal numbers.

8. **(Eng4P31)** Let a, b, c, d be integers. Show that the product of the six differences $b-a, c-a, d-a, c-b, d-b$, and $d-c$ is divisible by 12.

9. Five points lie in an equilateral triangle with side 1. Show that two of the points lie no farther than 1/2 apart. Can the "1/2" be replaced by anything smaller? Can it be improved if the "five" is replaced by "six"?

10. A lattice point in the plane is a point (x, y) such that both x and y are integers. Find the smallest number n such that given n lattice points in the plane, there exist two whose midpoint is also a lattice point.

11. Let u be an irrational real number. Let S be the set of all real numbers of the form $a + bu$, where a and b are integers. Show that S is dense in the real numbers, i.e. for any real number x and any $\varepsilon > 0$, there is an element $y \in S$ such that $|x - y| < \varepsilon$. (Hint: First let $x = 0$.)

12. **(1988B3)** For every n in the set $\mathbb{Z}^+ = \{1, 2, \ldots\}$ of positive integers, let r_n be the minimum value of $|c - d\sqrt{3}|$ for all nonnegative integers c and d with $c + d = n$. Find, with proof, the smallest positive real number g with $r_n \leq g$ for all $n \in \mathbb{Z}^+$.

13. Prove that there is some integral power of 2 that begins with 2004...

14. **(2006A3)** Let 1, 2, 3, ..., 2005, 2006, 2007, 2009, 2012, 2016, ... be a sequence defined by $x_k = k$ for $k = 1, 2, \ldots, 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.

15. Given any $n + 1$ integers between 1 and $2n$, show that one of them is divisible by another.

16. Show that if there are n people at a party, then two of them know the same number of people (among those present). It is assumed that "knowing" someone is a symmetric relationship: if A knows B, then B knows A.

17. Prove that in any group of six people there are either three mutual friends or three mutual strangers. (Hint: Represent the people by the vertices of a regular hexagon. Connect two vertices with a red line segment if the couple represented by these vertices are friends; otherwise, connect them with a blue line segment.)

18. Follow-up to the previous question: Find some n so that in any group of n people there are either four mutual friends or four mutual strangers.

19. A polygon in the plane has area 1.2432. Show that it contains two distinct points (x_1, y_1) and (x_2, y_2) that differ by (a, b), where a and b are integers.

20. **(2000B6)** Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(\pm 1, \pm 1, \ldots, \pm 1)$ in n-dimensional space with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.

21. **(2000B1)** Let a_j, b_j, c_j be integers for $1 \leq j \leq N$. Assume, for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \leq j \leq N$.

22. **(Aqu)** An infinite checkerboard is colored using a finite number of colors (i.e, each square of the board is colored in one of the several available colors). Show that it is always possible to pick four squares of the same color in such a way that they are the vertices of a rectangle with sides parallel to the sides of the squares.

23. **(1985B3)** Let

$$\begin{array}{cccc} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

be a doubly infinite array of positive integers, and suppose each positive integer appears exactly eight times in the array. Prove that $a_{m,n} > mn$ for some pair of positive integers (m, n).

24. **(1994A6)** Let $f_1, f_2, \dots, f_{10}$ be bijections of the set of integers such that for each integer n, there is some composition $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_m}$ of these functions (allowing repetitions) which maps 0 to n. Consider the set of 1024 functions

$$\mathcal{F} = \{f_1^{e_1} \circ f_2^{e_2} \circ \cdots \circ f_{10}^{e_{10}}\},$$

$e_i = 0$ or 1 for $1 \leq i \leq 10$. (f_i^0 is the identity function and $f_i^1 = f_i$.) Show that if A is any nonempty finite set of integers, then at most 512 of the functions in $\mathcal{F}$ map A to itself.

The principle of induction can be formulated as follows. For each positive integer n, let $P(n)$ denote a statement. We want to prove that all the infinitely many statements $P(n)$ for $n = 1, 2, \dots$ are true. To do this, it suffices to check that:

- *$P(1)$ is true;*
- *for all n, $P(n)$ implies $P(n+1)$.*

The next few problems are for practice with induction. Sometimes, we need to recast the problem a bit before induction can be applied.

25. Show that for all positive integers n, $n^5/5 + n^4/2 + n^3/3 - n/30$ is an integer.

26. Show that $1 + 1/\sqrt{2} + 1/\sqrt{3} + \cdots + 1/\sqrt{n} < 2\sqrt{n}$.

27. **(2005A1)** Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$.)

28. **(1992A1)** Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions:

    (i) $f(f(n)) = n$, for all integers n;

    (ii) $f(f(n+2)+2) = n$ for all integers n;

    (iii) $f(0) = 1$.

29. Prove that all even perfect squares are divisible by 4. Prove that all odd perfect squares leave a remainder of 1 upon division by 8. (This is a useful fact to know!) What are the possible remainders when you divide a perfect square by 3?

30. Consider the sequence of Fibonacci numbers, defined by $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. Prove that the following identities hold for all $n \geq 1$

    (a) $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$,

    (b)
    $$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix},$$

    (c) $F_{n+1}F_{n+2} - F_n F_{n+3} = (-1)^n$,

    (d) $F_1F_2 + F_2F_3 + \cdots + F_{2n-1}F_{2n} = F_{2n}^2$

    (e) $F_n^3 + F_{n+1}^3 - F_{n-1}^3 = F_{3n}$.

31. Let $a_1, a_2, \ldots, a_n$ be positive numbers. Show that
    $$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n},$$
    with equality only if $a_1 = a_2 = \cdots = a_n$. (This is the famous inequality between AM and GM).

32. **(TT1987)** Prove that, for all integers $n \geq 2$, the following inequality holds:
    $$\sqrt{2\sqrt{3\sqrt{4 \cdots \sqrt{(n-1)\sqrt{n}}}}} < 3.$$

33. **(1993A2)** Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$. Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

34. **(1990B2)** Prove that for $|x| < 1$, $|z| > 1$,

$$1 + \sum_{j=1}^{\infty} (1 + x^j) \frac{(1-z)(1-zx)(1-zx^2)\cdots(1-zx^{j-1})}{(z-x)(z-x^2)(z-x^3)\cdots(z-x^j)} = 0.$$

35. **(1990A1)** Let

$$T_0 = 2,\ T_1 = 3,\ T_2 = 6,$$

and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$

The first few terms are

$$2,\ 3,\ 6,\ 14,\ 40,\ 152,\ 784,\ 5168,\ 40576,\ 363392.$$

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where (A_n) and (B_n) are well-known sequences.

36. **(1985B2)** Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and

$$\frac{d}{dx}(f_{n+1}(x)) = (n+1)f_n(x+1)$$

for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

The method of induction is based on the fact that every non-empty set of positive integers has a smallest element. This method can be generalized as follows. Suppose we must prove that something is true for all objects of some class. We reason by contradiction: if it is not true for all, then there must be a counterexample. Pick the counterexample that is "minimal" in a certain sense (make sure that a minimal one must exist). Do something to it, and obtain a "smaller" counterexample, thus arriving at a contradiction... (A sure indication that the method of the minimal counterexample is called for is the following: it sounds like an induction problem, but at the last moment the induction step seems to be "slipping away".) Variation: sometimes the word "minimal" must be replaced by "maximal". The next few problems are for practice with the method of the minimal counterexample.

37. Show that there are infinitely many prime numbers.

38. **(Aqu12P41)** Is it possible to fill a box completely with finitely many cubes of different sizes?

39. **(Aqu5P2)** A group of $2k$ knights is gathering at a round table. Each one of them has at most $k-1$ enemies among the other knights. Show that it is possible to seat them around the table in such a way that no two enemies will be seated next to each other.

40. **(Aqu12P60)** Let f be a function of two variables such that the inequality $f(x,y) > 0$ describes the first quadrant, i.e.

$$\{(x,y) : f(x,y) > 0\} = \{(x,y) : x > 0, y > 0\}.$$

Show that f cannot be a polynomial.

41. **(Aqu12P43)** Let a and b be two fixed integers. Consider the following operation on the set of all integers:

$$m * n = \begin{cases} \dfrac{m+n}{2} + a, & \text{if } m+n \text{ is even,} \\ \dfrac{m+n+1}{2} + b, & \text{if } m+n \text{ is odd.} \end{cases}$$

Let M be the set of numbers that can be obtained by starting with 0, and using the operation $*$. Show that M is closed under addition (i.e., $m, n \in M$ implies $m + n \in M$).

42. **(Aqu5P3)** Show that, for all integers $n \geq 2$, the number

$$\sqrt{2} + \sqrt{3} + \cdots + \sqrt{n}$$

is irrational.

43. **(1993B6)** Let S be a set of three, not necessarily distinct, positive integers. Show that one can transform S into a set containing 0 by a finite number of applications of the following rule: Select two of the three integers, say x and y, where $x < y$ and replace them with $2x$ and $y - x$.

44. **(Eng8p39)** Let x_1 and x_2 be the roots of the equation $x^2 + px - 1 = 0$, where p is an odd integer. For $n \geq 0$, define $y_n = x_1^n + x_2^n$. Prove that, for all integers $n \geq 0$, the numbers y_n and y_{n+1} are co-prime integers (i.e., integers with no common factors).

45. **(2001B4)** Let S denote the set of rational numbers different from $\{-1, 0, 1\}$. Define $f : S \to S$ by $f(x) = x - 1/x$. Prove or disprove that

$$\bigcap_{n=1}^{\infty} f^{(n)}(S) = \emptyset,$$

where $f^{(n)}$ denotes f composed with itself n times.