ELIZABETHTOWN COLLEGE
Putnam Preparation Series
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THEME 3

1. TOPICS

There are some problems whose solution, although not immediately obvious, becomes natural if we introduce convenient notation, reinterpret the problem, or make use of symmetry. These are the types of problems that we will be looking at today.

2. PRACTICE PROBLEMS

The first group of problems involve symmetry in one way or another. In situations like this we can use the symmetry to reduce the problem to an easier (although non-symmetric) problem. On some rare ocasions, the symmetry in the problem is a "red herring"; it helps make the problem work, but it does not help in solving it. One or more of the problems below are like this (and I'm not telling which).

1. Solve the system
$$\begin{cases} x^2+xy+y^2 &= 13 \\ x^2y^2+xy &= 12 \end{cases}$$

2. **(1998B2)** Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the x-axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

3. **(UDN001)** Solve the following equation for x:
$$\frac{(x^2-x+1)^3}{x^2(x-1)^2} = \frac{(\pi^2-\pi+1)^3}{\pi^2(\pi-1)^2}$$

4. **(UDN026)** Find all the solutions of the system
$$\begin{cases} \cos x_1 &= x_2 \\ \cos x_2 &= x_3 \\ \cdots & \cdots \\ \cos x_{98} &= x_{99} \\ \cos x_{99} &= x_1 \end{cases}$$

5. **(1998B1)** Find the minimum value of
$$\frac{(x+1/x)^6-(x^6+1/x^6)-2}{(x+1/x)^3+(x^3+1/x^3)}$$
for $x > 0$.

6. **(UDN028)** The real numbers $a_1, a_2, \ldots, a_n$ satisfy the inequalities

$$\begin{cases} a_1 - 2a_2 + a_3 & \geq \ 0 \\ a_2 - 2a_3 + a_4 & \geq \ 0 \\ \cdots & \ \ \cdots \\ a_{n-2} - 2a_{n-1} + a_n & \geq \ 0 \\ a_{n-1} - 2a_n + a_1 & \geq \ 0 \\ a_n - 2a_1 + a_2 & \geq \ 0 \end{cases}$$

Show that $a_1 = a_2 = \cdots = a_n$

7. **(1997A5)** Let N_n denote the number of ordered n-tuples of positive integers $(a_1, a_2, \ldots, a_n)$ such that $1/a_1 + 1/a_2 + \ldots + 1/a_n = 1$. Determine whether N_{10} is even or odd.

8. **(1988B1)** A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \ldots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y, z positive integers.

9. **(1993A5)** Show that

$$\int_{-100}^{-10} \left(\frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx + \int_{\frac{1}{101}}^{\frac{1}{11}} \left(\frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx + \int_{\frac{101}{100}}^{\frac{11}{10}} \left(\frac{x^2 - x}{x^3 - 3x + 1} \right)^2 dx$$

is a rational number.

10. Compute the integrals **(Chl01,Chl01,1980A3,1987B1,...)**:

(a) $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1 + e^x}\, dx$ **(b)** $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$

(c) $\displaystyle\int_0^{\pi/2} \frac{1}{1 + (\tan x)^{\sqrt{2}}}\, dx$ **(d)** $\displaystyle\int_2^4 \frac{\sqrt{\ln(9 - x)}}{\sqrt{\ln(9 - x)} + \sqrt{\ln(x + 3)}}\, dx$

(e) $\displaystyle\int_0^{\infty} \frac{\ln x}{1 + x^2}\, dx$ **(f)** $\displaystyle\int_0^1 \ln \sin \frac{\pi x}{2}\, dx$

11. **(1985A1)** Let k be the smallest positive integer with the following property:

 There are distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial

 $$p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)$$

 has exactly k nonzero coefficients.

 Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

The second group consists of problems that look hard, but become much easier if the right notation, interpretation, or representation is found. A cumbersome algebraic problem can become obvious if represented geometrically, complicated relationships become easy to grasp if represented by a graph, and painting some numbers green or red can sometimes give an unexpected burst of insight.

12. **(Aqu8P1)** Given six people, prove that it is always possible to chose three of them in such a way that either every two of the three know each other, or every two of the three don't know each other.

13. **(Aqu8P6)** A countably infinite number of bags are labeled with the positive integers $1, 2, \ldots$ etc. In each bag there is a card with a positive integer, *different* from the number on the label. Show that it is possible to chose infinitely many bags, so that the numbers on the cards in these bags are different from all the numbers on the labels of these bags.

14. **(Aqu8P9)** At a reception, a large number of men and women are present. It turns out that each man knows exactly 10 of the women, and each woman knows exactly 10 of the men. Show that there is the same number of men and women at this reception.

15. **(2004A5)** An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$. We say that two squares, p and q, are in the same connected monochromatic component if there is a sequence of squares, all of the same color, starting at p and ending at q, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$.

16. **(1985A1)** Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that

    (i) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and

    (ii) $A_1 \cap A_2 \cap A_3 = \emptyset$,

    where $\emptyset$ denotes the empty set. Express the answer in the form $2^a 3^b 5^c 7^d$, where a, b, c, and d are nonnegative integers.

17. **(2003A6)** For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S$, $s_2 \in S$, $s_1 \neq s_2$, and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n?

18. **(Aqu10P5)** All the five vertices of a pentagon have integer coordinates on the plane, and all the sides of the pentagon have integer lengths. Show that its perimeter is an even number.

19. **(Aqu11P6)** The positive real numbers x, y, and z satisfy the equalities

$$\begin{cases} x^2 + \sqrt{3}xy + y^2 &= 25 \\ y^2 + z^2 &= 9 \\ z^2 + zx + x^2 &= 16 \end{cases}$$

Compute the value of the expression $xy + 2yz + \sqrt{3}zx$.

20. **(1994A2)** Let A be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the x-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the y-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$.

21. **(1995B2)** An ellipse, whose semi-axes have lengths a and b, rolls without slipping on the curve $y = c\sin\left(\frac{x}{a}\right)$. How are a, b, c related, given that the ellipse completes one revolution when it traverses one period of the curve?

22. **(1996A1)** Find the least number A such that for any two squares of combined area 1, a rectangle of area A exists such that the two squares can be packed in the rectangle (without the interiors of the squares overlapping). You may assume that the sides of the squares will be parallel to the sides of the rectangle.

23. **(2001B1)** Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid so that the k-th row, from left to right, is

$$(k-1)n + 1, (k-1)n + 2, \ldots, (k-1)n + n.$$

Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.