ELIZABETHTOWN COLLEGE
Putnam Preparation Series
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THEME 4

1. TOPICS

Today I will focus on two loosely related topics: linear recursions and Linear Algebra.

Let $a_0, a_1, a_2, \ldots, a_n, \ldots$ be a sequence that satisfies the recursive relation

$$a_n = pa_{n-1} + qa_{n-2},$$

with a_0 and a_1 given. A general formula for the n^{th} term can be found as follows. Write the characteristic equation

$$\lambda^2 - p\lambda - q = 0,$$

and let λ_1 and λ_2 be its roots. Then, if $\lambda_1 \neq \lambda_2$,

$$a_n = C_1\lambda_1^n + C_2\lambda_2^n,$$

and if $\lambda_1 = \lambda_2$, then

$$a_n = C_1\lambda_1^n + C_2 n\lambda_1^n.$$

The constants C_1 and C_2 are determined from a_0 and a_1.

This rule can be proved easily by induction. It generalizes naturally for recursions of higher order.

Some of the problems below are stolen from Ravi Vakil's Putnam website at Stanford.

2. PRACTICE PROBLEMS

The first problems are for practice with linear recursion.

1. The sequence $q_0, q_1, q_2, \ldots$ satisfies $q_n = 3q_{n-2} - 2q_{n-3}$, and $q_0 = 0$, $q_1 = 1$, $q_2 = 11$. Find a general formula for q_n .

2. Compute
$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n.$$

3. The sequence $r_1, r_2, \ldots$ satisfies $r_n = (5/2)r_{n-1} - r_{n-2}$, and $r_1 = 2004$. Suppose the sequence converges to a finite real number. Find r_2.

4. The sequence $G_0, G_1, G_2, \ldots$ consists of every other Fibonacci number. Show that there exists a linear recursion of the form $G_n = aG_{n-1} + bG_{n-2}$. (Follow-up: How about a sequence consisting of every tenth Fibonacci number. How do you know there's a recursion? With integer coefficients?)

5. Use the theory of linear recursive sequences to find a formula for the sequence $s_0 = 1, s_1 = 2, s_n = s_{n-2}$. What do you observe? Now try a sequence with period four, such as $t_0 = 1, t_1 = 0, t_2 = 0, t_3 = 0, t_n = t_{n-4}$.

6. Let $I_n = \int_0^{\pi/2} \sin^n x\, dx$. Find a recurrence relation for I_n. Use this relation to show that

$$I_{2n} = \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2 \times 4 \times 6 \times \cdots \times (2n)} \cdot \frac{\pi}{2},$$

and

$$I_{2n+1} = \frac{2 \times 4 \times 6 \times \cdots \times (2n-2)}{1 \times 3 \times 5 \times \cdots \times (2n-1)}.$$

Write these formulas in terms of factorials.

7. **(1984B6)** A sequence of convex polygons (P_n), $n \geq 0$, is defined inductively as follows. P_0 is an equilateral triangle with sides of length 1. Once P_n has been determined, its sides are trisected; the vertices of P_{n+1} are the interior trisection points of the sides of P_n. Thus P_{n+1} is obtained by cutting corners of P_n, and P_n has $3 \cdot 2^n$ sides. (P_1 is a regular hexagon with sides of length 1/3.) Express $\lim_{n\to\infty} \text{Area}(P_n)$ in the form $\sqrt{a}/b$, where a and b are positive integers.

Recursions in Probability.

8. Two ping pong players, A and B, agree to play several games. The players are evenly matched; suppose, however, that whoever serves first has probability p of winning that game (this may be player A in one game, or player B in another). Suppose A serves first in the first game, but thereafter the loser serves first. Let p_n denote the probability that A wins the nth game. Show that $p_{n+1} = p_n(1-p) + (1-p_n)p$. If p is neither 0 nor 1, you might expect that the limit of p_n is 1/2. Why? Can you prove this?

9. A gambling graduate student tosses a fair coin and scores one point for each head that turns up and two points for each tail. Prove that the probability of the student scoring exactly n points at some time in a sequence of n tosses is $(2+(-1/2)^n)/3$. (Hint: Let p_n denote the probability of scoring exactly n points at some time. Express p_n in terms of p_{n-1} or in terms of p_{n-1} and p_{n-2}. Use this linear recursion to give an inductive proof. Even better hint, useful in many circumstances: you've been given the answer, so reverse-engineer the recursion, and then try to prove it.)

Recursions in Determinants.

10. Calculate Vandermonde's determinant

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}.$$

11. This problem deals with determinants of matrices of some special form.

    (a) Let E_n denote the determinant of the $n \times n$ matrix having -1's below the main diagonal (from upper left to lower right) and 1's on and above the main diagonal. Show that $E_1 = 1$, and $E_n = 2E_{n-1}$ for $n > 1$.

(b) Let D_n denote the determinant of the $n \times n$ matrix whose (i, j)th entry (the element of the ith row and jth column) is the absolute value of the difference of i and j. Show that

$$D_n = (-1)^{n-1}(n-1)2^{n-2}.$$

(c) Evaluate the $n \times n$ determinant A_n whose (i, j)th entry is $a^{|i-j|}$ by finding a recursive relationship between A_n and A_{n-1}.

(d) Let F_n denote the determinant of the $n \times n$ matrix with a on the main diagonal, b on the superdiagonal (the diagonal immediately above the main diagonal – having $n-1$ entries), and c on the subdiagonal (the diagonal immediately below the main diagonal – having $n-1$ entries). Show that $F_n = aF_{n-1} - bcF_{n-2}$, for $n > 2$. What happens when $a = b = 1$ and $c = -1$?

12. Compute the $n \times n$ determinants:

$$\text{(a)} \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & a_1 & 0 & \cdots & 0 \\ 1 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & a_{n-1} \end{vmatrix} \qquad \text{(b)} \begin{vmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \cdots & 0 \\ 0 & 1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{vmatrix}$$

$$\text{(c)} \begin{vmatrix} 3 & 2 & 0 & \cdots & 0 \\ 1 & 3 & 2 & \cdots & 0 \\ 0 & 1 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 3 \end{vmatrix} \qquad \text{(d)} \begin{vmatrix} a+b & ab & 0 & \cdots & 0 \\ 1 & a+b & ab & \cdots & 0 \\ 0 & 1 & a+b & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a+b \end{vmatrix}$$

13. **(1992B5)** Let D_n denote the value of the $(n-1) \times (n-1)$ determinant

$$\begin{vmatrix} 3 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 4 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 5 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 6 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \ldots & n+1 \end{vmatrix}$$

Is the set $\{D_n/n!\}_{n \geq 2}$ bounded?

Linear Operators and Matrices.

14. A matrix (m_{ij}) is circulant if the entry m_{ij} depends only on $j-i$ modulo n. Find the eigenvectors of a circulant $n \times n$ matrix. (Hint: Try the case $n = 2$, and make a guess!)

15. For which values of n does there exist an $n \times n$ (real) matrix A such that $A^2 = -I_n$? Here I_n denotes the $n \times n$ identity matrix.

16. For which values of n do there exist two $n \times n$ matrices A and B such that $AB - BA = I_n$? Here again I_n is the $n \times n$ identity matrix.

17. **(2005A4)** Let H be an $n \times n$ matrix all of whose entries are ± 1 and whose rows are mutually orthogonal. Suppose H has an $a \times b$ submatrix whose entries are all 1. Show that $ab \leq n$.

18. Let $n \geq 1$ and let A and B be $n \times n$ matrices such that the matrix $(I_n - AB)$ is invertible. Prove that the matrix $(I_n - BA)$ is also invertible. Again, I_n denotes the $n \times n$ identity matrix.

19. **(1990A5)** (modified) In this problem 0 denotes the zero matrix of appropriate size.

    (a) Let A and B be two 2×2 matrices such that $ABAB = 0$. Does it follow that $BABA = 0$?

    (b) Same question, if A and B are 3×3 matrices.

    (c) For what values of the integers $n \geq 1$ and $m \geq 1$ do there exist $n \times n$ matrices A and B, such that $(AB)^m = 0$, but $(BA)^m \neq 0$?

20. **(1994B4)** For $n \geq 1$, let d_n be the greatest common divisor of the entries of $A^n - I$, where

    $$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

    Show that $\lim_{n \to \infty} d_n = \infty$.

21. **(1988A6)** If a linear transformation A on an n-dimensional vector space has $n + 1$ eigenvectors such that any n of them are linearly independent, does it follow that A is a scalar multiple of the identity? Prove your answer.

22. **(1996B4)** For any square matrix A, we can define $\sin A$ by the usual power series:

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$

Prove or disprove: there exists a 2×2 matrix A with real entries such that

$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}.$$

23. **(1987B5)** Let O_n be the n-dimensional vector $(0, 0, \cdots, 0)$. Let M be a $2n \times n$ matrix of complex numbers such that whenever $(z_1, z_2, \ldots, z_{2n})M = O_n$, with complex z_i, not all zero, then at least one of the z_i is not real. Prove that for arbitrary real numbers $r_1, r_2, \ldots, r_{2n}$, there are complex numbers $w_1, w_2, \ldots, w_n$ such that

$$\operatorname{Re}\left[M \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \right] = \begin{pmatrix} r_1 \\ \vdots \\ r_{2n} \end{pmatrix}.$$

(Note: if C is a matrix of complex numbers, $\operatorname{Re}(C)$ is the matrix whose entries are the real parts of the entries of C.)

24. **(1986B6)** Suppose A, B, C, D are $n \times n$ matrices with entries in a field F, satisfying the conditions that AB^t and CD^t are symmetric and $AD^t - BC^t = I$. Here I is the $n \times n$ identity matrix, and if M is an $n \times n$ matrix, M^t is its transpose. Prove that $A^tD - C^tB = I$.

25. **(1985B6)** Let G be a finite set of real $n \times n$ matrices $\{M_i\}$, $1 \leq i \leq r$, which form a group under matrix multiplication. Suppose that $\sum_{i=1}^{r} \operatorname{tr}(M_i) = 0$, where $\operatorname{tr}(A)$ denotes the trace of the matrix A. Prove that $\sum_{i=1}^{r} M_i$ is the $n \times n$ zero matrix.