ELIZABETHTOWN COLLEGE
Putnam Preparation Series
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THEME 5

1. TOPICS

Today we will spend some time on that part of Calculus and Analysis that deals with sequences and series.

We will not talk about linear recursions, because we talked about them last week. We won't talk about generating functions either, because I am planning a separate session for those.

2. PRACTICE PROBLEMS

1. **(Chl12)** Find the limit

$$\lim_{n\to\infty}\sum_{k=0}^{16}\sqrt{n+k}\,\cos\frac{2k\pi}{17}.$$

2. **(Chl13)** Is it true that every sequence of real numbers contains a monotonic (i.e. non-increasing or non-decreasing) subsequence?

3. **(Chl14)** Is it true that every convergent sequence of real numbers can be represented as the difference of two nondecreasing bounded sequences?

4. **(1985A3)** Let d be a real number. For each integer $m \geq 0$, define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \ldots$ by the condition

$$a_m(0) = d/2^m, \qquad \text{and} \qquad a_m(j+1) = (a_m(j))^2 + 2a_m(j), \text{ for } j \geq 0.$$

Evaluate $\lim_{n\to\infty} a_n(n)$.

5. **(1993A2)** Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$. Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

6. **(1992B3)** For any pair (x, y) of real numbers, a sequence $(a_n(x, y))_{n\geq 0}$ is defined as follows:

$$\begin{aligned} a_0(x, y) &= x, \\ a_{n+1}(x, y) &= \frac{(a_n(x, y))^2 + y^2}{2}, \qquad \text{for } n \geq 0. \end{aligned}$$

Find the area of the region

$$\{(x, y)|(a_n(x, y))_{n\geq 0} \quad \text{converges}\}.$$

7. **(Chl08)** Let $x \in (0, 1)$ and define two sequences of numbers, (a_n), (b_n), $n = 0, 1, 2, \ldots$ by the following recursive procedure:

$$\begin{aligned} a_0 &= x, \qquad & a_{n+1} &= \sin a_n, \\ b_0 &= x, \qquad & b_{n+1} &= \frac{b_n}{\sqrt{1 + b_n^2/3}}. \end{aligned}$$

(a) Prove that $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist and are equal to 0.
(b) Find real numbers $\alpha > 0$, $0 < L < \infty$, such that

$$\lim_{n\to\infty} n^\alpha b_n = L.$$

What can you say about

$$\lim_{n\to\infty} n^\alpha a_n?$$

8. **(1987B4)** Let $(x_1, y_1) = (0.8, 0.6)$ and let $x_{n+1} = x_n \cos y_n - y_n \sin y_n$ and $y_{n+1} = x_n \sin y_n + y_n \cos y_n$ for $n = 1, 2, 3, \ldots$. For each of $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$, prove that the limit exists and find it or prove that the limit does not exist.

9. **(2004A3)** Define a sequence $\{u_n\}_{n=0}^\infty$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all $n \geq 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)

10. Let (a_n) be a sequence of nonnegative reals such that, for all n and m, $a_{n+m} \leq a_n + a_m$. Prove that the sequence (a_n/n) converges. What can you say about its limit?

11. **(2001B6)** Assume that $(a_n)_{n\geq 1}$ is an increasing sequence of positive real numbers such that $\lim a_n/n = 0$. Must there exist infinitely many positive integers n such that $a_{n-i} + a_{n+i} < 2a_n$ for $i = 1, 2, \ldots, n-1$?

12. **(1994A1)** Suppose that a sequence $a_1,\ a_2,\ a_3,\ \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

13. **(2000A1)** Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, \ldots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

14. **(Chl15)** Does the following series converge:
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n} - (-1)^n} \quad ?$$

15. **(Chl16)** Let (a_n) and (b_n), $n = 0, 1, 2, \ldots$ be two increasing sequences of positive numbers such that the series
$$\sum_{n=1}^{\infty} \frac{1}{a_n} \text{ and } \sum_{n=1}^{\infty} \frac{1}{b_n}$$
both diverge. Could it be that the series
$$\sum_{n=1}^{\infty} \frac{1}{a_n + b_n}$$
converges?

16. **(Chl17)** Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. (Here a_n are real numbers, not necessarily positive.) Is it possible that the series $\sum_{n=1}^{\infty} a_n^3$ diverges?

17. **(1988B4)** Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is $\sum_{n=1}^{\infty} (a_n)^{n/(n+1)}$.

18. **(Chl10)** Let $x_0 = 0$ and $x_{n+1} = \sqrt{1 + x_n}$ for $n \geq 1$. What can you say about the convergence of the series

$$\sum_{n=1}^{\infty} \left(1 - \frac{x_n}{x_{n+1}}\right)?$$

19. **(1988A3)** Determine, with proof, the set of real numbers x for which

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \csc \frac{1}{n} - 1\right)^x$$

converges.

20. **(1987A6)** For each positive integer n, let $a(n)$ be the number of zeroes in the base 3 representation of n. For which positive real numbers x does the series

$$\sum_{n=1}^{\infty} \frac{x^{a(n)}}{n^3}$$

converge?

21. Compute the sums:

$$(\mathbf{a}) \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)} \qquad (\mathbf{b})(\mathbf{1986A3}) \sum_{n=1}^{\infty} \operatorname{arccot}(n^2 + n + 1)$$

$$(\mathbf{c})(\mathbf{RB26}) \sum_{n=1}^{\infty} \arctan\left(\frac{2}{n^2}\right)$$

22. **(1999A4)** Sum the series

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m (n 3^m + m 3^n)}.$$

23. **(1984A2)** Express

$$\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$$

as a rational number.

24. **(2001B3)** For any positive integer n, let $\langle n\rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n\rangle} + 2^{-\langle n\rangle}}{2^n}.$$

25. **(Chl18)** Let (a_n), $n = 1, 2, 3, \ldots$ be a sequence of positive integers such that $a_1 \geq 2$ and $a_n \leq a_{n+1}$ for all n. Show that the number

$$\alpha := \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \ldots a_n}$$

is rational if and only if there exists a positive integer k such that $a_n = a_k$ for all $n \geq k$.