ELIZABETHTOWN COLLEGE
Putnam Preparation Series
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# THEME 6

## 1. TOPICS

Today I will talk about combinatorics, power series and generating functions, and probability.

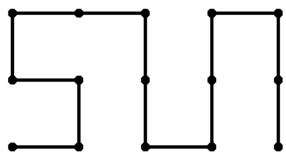
## 2. PRACTICE PROBLEMS

*The first problems are for practice with Combinatorics and counting.*

1. **(2005A2)** Let $\mathcal{S} = \{(a, b)|a = 1, 2, \ldots, n, b = 1, 2, 3\}$. A *rook tour* of **S** is a polygonal path made up of line segments connecting points $p_1, p_2, \ldots, p_{3n}$ in sequence such that

   (i) $p_i \in \mathcal{S}$,
   (ii) $p_i$ and $p_{i+1}$ are a unit distance apart, for $1 \leq i < 3n$,
   (iii) for each $p \in \mathcal{S}$ there is a unique $i$ such that $p_i = p$.

   How many rook tours are there that begin at $(1, 1)$ and end at $(n, 1)$? (An example of such a rook tour for $n = 5$ is depicted.)



2. **(2004B2)** Let $m$ and $n$ be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$

3. **(2002A3)** Let $n \geq 2$ be an integer and $T_n$ be the number of non-empty subsets $S$ of $\{1, 2, 3, \ldots, n\}$ with the property that the average of the elements of $S$ is an integer. Prove that $T_n - n$ is always even.

4. **(1993A3)** Let $\mathcal{P}_n$ be the set of subsets of $\{1, 2, \ldots, n\}$. Let $c(n, m)$ be the number of functions $f : \mathcal{P}_n \to \{1, 2, \ldots, m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that

$$c(n, m) = \sum_{j=1}^{m} j^n.$$

5. **(1996A3)** Suppose that each of 20 students has made a choice of anywhere from 0 to 6 courses from a total of 6 courses offered. Prove or disprove: there are 5 students and 2 courses such that all 5 have chosen both courses or all 5 have chosen neither course.

6. **(1996B1)** Define a *selfish* set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.

7. **(1990A6)** If $X$ is a finite set, let $|X|$ denote the number of elements in $X$. Call an ordered pair $(S, T)$ of subsets of $\{1, 2, \ldots, n\}$ *admissible* if $s > |T|$ for each $s \in S$, and $t > |S|$ for each $t \in T$. How many admissible ordered pairs of subsets of $\{1, 2, \ldots, 10\}$ are there? Prove your answer.

8. **(1995B1)** For a partition $\pi$ of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, let $\pi(x)$ be the number of elements in the part containing $x$. Prove that for any two partitions $\pi$ and $\pi'$, there are two distinct numbers $x$ and $y$ in $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A *partition* of a set $S$ is a collection of disjoint subsets (parts) whose union is $S$.]

9. **(1995A6)** Suppose that each of $n$ people writes down the numbers 1,2,3 in random order in one column of a $3 \times n$ matrix, with all orders equally likely and with the orders for different columns independent of each other. Let the row sums $a, b, c$ of the resulting matrix be rearranged (if necessary) so that $a \leq b \leq c$. Show that for some $n \geq 1995$, it is at least four times as likely that both $b = a + 1$ and $c = a + 2$ as that $a = b = c$.

10. **(1996B5)** Given a finite string $S$ of symbols $X$ and $O$, we write $\Delta(S)$ for the number of $X$'s in $S$ minus the number of $O$'s. For example, $\Delta(XOOXOOX) = -1$. We call a string $S$ *balanced* if every substring $T$ of (consecutive symbols of) $S$ has $-2 \leq \Delta(T) \leq 2$. Thus, $XOOXOOX$ is not balanced, since it contains the substring $OOXOO$. Find, with proof, the number of balanced strings of length $n$.

11. **(1986A4)** A *transversal* of an $n \times n$ matrix $A$ consists of $n$ entries of $A$, no two in the same row or column. Let $f(n)$ be the number of $n \times n$ matrices $A$ satisfying the following two conditions:

    (a) Each entry $\alpha_{i,j}$ of $A$ is in the set $\{-1, 0, 1\}$.

    (b) The sum of the $n$ entries of a transversal is the same for all transversals of $A$.

    An example of such a matrix $A$ is

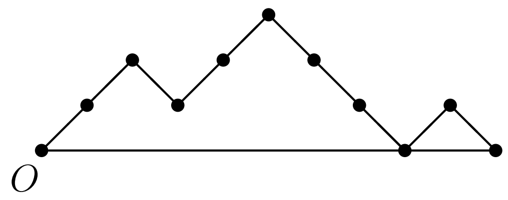
$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

    Determine with proof a formula for $f(n)$ of the form

$$f(n) = a_1 b_1^n + a_2 b_2^n + a_3 b_3^n + a_4,$$

    where the $a_i$'s and $b_i$'s are rational numbers.

12. **(2003A5)** A Dyck $n$-path is a lattice path of $n$ upsteps $(1, 1)$ and $n$ downsteps $(1, -1)$ that starts at the origin $O$ and never dips below the $x$-axis. A return is a maximal sequence of contiguous downsteps that terminates on the $x$-axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.




    Show that there is a one-to-one correspondence between the Dyck $n$-paths with no return of even length and the Dyck $(n-1)$-paths.

*Power series.*

13. **(1992A2)** Define $C(\alpha)$ to be the coefficient of $x^{1992}$ in the power series about $x = 0$ of $(1+x)^\alpha$. Evaluate

$$\int_0^1 \left( C(-y-1) \sum_{k=1}^{1992} \frac{1}{y+k} \right) dy.$$

14. **(1998B5)** Let $N$ be the positive integer with 1998 decimal digits, all of them 1; that is,

$$N = 1111 \cdots 11.$$

Find the thousandth digit after the decimal point of $\sqrt{N}$.

15. **(Chl06)** Assuming that $\tan x$ has the representation

$$\tan x = c_0 x + c_1 x^3 + c_2 x^5 + \ldots = \sum_{n=0}^{\infty} c_n x^{2n+1}$$

in a neighborhood of 0, show that the coefficients can be obtained by the following recursive procedure:

$$c_0 = 1, \qquad c_{n+1} = \frac{1}{2n+3} \sum_{k=0}^{n} c_k c_{n-k}.$$

16. **(1999A3)** Consider the power series expansion

$$\frac{1}{1-2x-x^2} = \sum_{n=0}^{\infty} a_n x^n.$$

Prove that, for each integer $n \geq 0$, there is an integer $m$ such that

$$a_n^2 + a_{n+1}^2 = a_m.$$

17. **(1999B3)** Let $A = \{(x,y) : 0 \leq x, y < 1\}$. For $(x,y) \in A$, let

$$S(x,y) = \sum_{\frac{1}{2} \leq \frac{m}{n} \leq 2} x^m y^n,$$

where the sum ranges over all pairs $(m,n)$ of positive integers satisfying the indicated inequalities. Evaluate

$$\lim_{(x,y)\to(1,1),(x,y)\in A} (1-xy^2)(1-x^2y)S(x,y).$$

18. **(1997A3)** Evaluate

$$\int_0^\infty \left(x - \frac{x^3}{2} + \frac{x^5}{2\cdot 4} - \frac{x^7}{2\cdot 4\cdot 6} + \cdots\right)\left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} + \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots\right)dx.$$

19. Suppose

$$x = 0.12345\ldots = \sum_{k=1}^\infty \frac{k}{10^k}.$$

What is the thousandth digit of $x$ after the decimal place? Show that $x$ is rational. Find it.

20. Show that

$$0.0001\,0016\,0081\,0256\ldots = \sum_{k=1}^\infty \frac{k^4}{10^{4k}}$$

is a rational number. (Hint: Don't try to find it!)

*Generating Functions.*

21. Suppose $p(x) = (1+x+x^2)^{2001}$ is expanded out as a huge degree 4002 polynomial

$$p(x) = a_0 + a_1 x + \cdots + a_{4001}x^{4001} + a_{4002}x^{4002}.$$

(a) Find the sum of the coefficients of $p(x)$.
(b) Find the sum of every other coefficient, starting with $a_0$.
(c) Find the sum of every third coefficient.

22. Notice that

$$e^{ax}e^{bx} = e^{(a+b)x}.$$

Consider both sides as power series. Write down the coefficient of $x^n$ on each side. What equality have you just proved?

23. Let $a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 3, \ldots$ be the Fibonacci sequence, with $a_{n+2} = a_{n+1} + a_n$. Show that

$$\sum_{n=1}^\infty \frac{a_n}{4^{n+1}} = \frac{1}{11}.$$

24. Define the integer sequence $(T_n)$ by $T_0 = 0$, $T_1 = 1$, $T_2 = 2$, and $T_{n+1} = T_n + T_{n-1} + T_{n-2}$ for $n \geq 2$. Compute

$$S = \sum_{n=0}^{\infty} \frac{T_n}{2^n}.$$

25. **(UDN187)** Let

$$f(x) = 1 + \frac{x^2}{3} + \frac{x^4}{5} + \frac{x^6}{7} + \cdots, \qquad \text{for } |x| < 1.$$

Show that

$$f\left(\frac{2x}{1+x^2}\right) = (1+x^2)f(x).$$

26. Let $f_0(x) = e^x$ and $f_n(x) = xf'_{n-1}(x)$ for $n > 0$. Show that

$$\sum_{n=0}^{\infty} \frac{f_n(1)}{n!} = e^e.$$

27. Find the $100^{\text{th}}$ derivative at the origin, $f^{(100)}(0)$ of

$$f(x) = \frac{1}{x^2 + 3x + 2}.$$

28. **(1987A2)** The sequence of digits

$$123456789101112131415161718192021\ldots$$

is obtained by writing the positive integers in order. If the $10^n$-th digit in this sequence occurs in the part of the sequence in which the $m$-digit numbers are placed, define $f(n)$ to be $m$. For example, $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$.

29. **(1987B2)** Let $r, s$ and $t$ be integers with $0 \leq r$, $0 \leq s$ and $r + s \leq t$. Prove that

$$\frac{\binom{s}{0}}{\binom{t}{r}} + \frac{\binom{s}{1}}{\binom{t}{r+1}} + \cdots + \frac{\binom{s}{s}}{\binom{t}{r+s}} = \frac{t+1}{(t+1-s)\binom{t-s}{r}}.$$

30. **(1992B2)** For nonnegative integers $n$ and $k$, define $Q(n,k)$ to be the coefficient of $x^k$ in the expansion of $(1+x+x^2+x^3)^n$. Prove that

$$Q(n,k) = \sum_{j=0}^{k} \binom{n}{j}\binom{n}{k-2j},$$

where $\binom{a}{b}$ is the standard binomial coefficient. (Reminder: For integers $a$ and $b$ with $a \geq 0$, $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ for $0 \leq b \leq a$, with $\binom{a}{b} = 0$ otherwise.)

31. **(1991B4)** Suppose $p$ is an odd prime. Prove that

$$\sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

32. **(1997A6)** For a positive integer $n$ and any real number $c$, define $x_k$ recursively by $x_0 = 0$, $x_1 = 1$, and for $k \geq 0$,

$$x_{k+2} = \frac{cx_{k+1} - (n-k)x_k}{k+1}.$$

Fix $n$ and then take $c$ to be the largest value for which $x_{n+1} = 0$. Find $x_k$ in terms of $n$ and $k$, $1 \leq k \leq n$.

33. **(1997B4)** Let $a_{m,n}$ denote the coefficient of $x^n$ in the expansion of $(1+x+x^2)^m$. Prove that for all [integers] $k \geq 0$,

$$0 \leq \sum_{i=0}^{\lfloor \frac{2k}{3} \rfloor} (-1)^i a_{k-i,i} \leq 1.$$

*Probability.*

34. **(1989A4)** If $\alpha$ is an irrational number, $0 < \alpha < 1$, is there a finite game with an honest coin such that the probability of one player winning the game is $\alpha$? (An honest coin is one for which the probability of heads and the probability of tails are both $\frac{1}{2}$. A game is finite if with probability 1 it must end in a finite number of moves.)

35. **(1989B6)** Let $(x_1,\, x_2,\, \dots\, x_n)$ be a point chosen at random from the $n$-dimensional region defined by $0 < x_1 < x_2 < \cdots < x_n < 1$. Let $f$ be a continuous function on $[0,1]$ with $f(1) = 0$. Set $x_0 = 0$ and $x_{n+1} = 1$. Show that the expected value of the Riemann sum

$$\sum_{i=0}^{n} (x_{i+1} - x_i) f(x_{i+1})$$

is $\int_0^1 f(t)P(t)\,dt$, where $P$ is a polynomial of degree $n$, independent of $f$, with $0 \le P(t) \le 1$ for $0 \le t \le 1$.

36. **(2002B1)** Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

37. **(1985B4)** Let $C$ be the unit circle $x^2 + y^2 = 1$. A point $p$ is chosen randomly on the circumference $C$ and another point $q$ is chosen randomly from the interior of $C$ (these points are chosen independently and uniformly over their domains). Let $R$ be the rectangle with sides parallel to the $x$ and $y$-axes with diagonal $pq$. What is the probability that no point of $R$ lies outside of $C$?

38. **(2001A2)** You have coins $C_1, C_2, \dots, C_n$. For each $k$, $C_k$ is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the $n$ coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of $n$.

39. If you roll two dice, the sum of the points $S$ can be any integer between 2 and 12, inculsive. These are 11 distinct possibilities, but they are not equiprobable. Thus, $S = 7$ with probability $1/6$, but $S = 2$ only with probability $1/36$. Suppose that we can bias each die at will, so that, for each choice of probabilities $(p_1, p_2, p_3, p_4, p_5, p_6)$, the die will turn up $k$ points with probability $p_k$, for $k = 1, \dots, 6$. Is it possible to bias both dice in such a way that all the 11 possibilities for the sum $S$ become equiprobable? (It is permissible to bias each die in a different way.)

40. **(1992A6)** Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)

41. **(2006A4)** Let $S = \{1, 2, \ldots, n\}$ for some integer $n > 1$. Say a permutation $\pi$ of $S$ has a local maximum at $k \in S$ if

    (i) $\pi(k) > \pi(k+1)$ for $k = 1$;

    (ii) $\pi(k-1) < \pi(k)$ and $\pi(k) > \pi(k+1)$ for $1 < k < n$;

    (iii) $\pi(k-1) < \pi(k)$ for $k = n$.

    (For example, if $n = 5$ and $\pi$ takes values at $1, 2, 3, 4, 5$ of $2, 1, 4, 5, 3$, then $\pi$ has a local maximum of 2 at $k = 1$, and a local maximum of 5 at $k = 4$.) What is the average number of local maxima of a permutation of $S$, averaging over all permutations of $S$?

42. Suppose $A$ and $B$ are candidates for office, and there are $a$ voters voting for $A$, $b$ voters for $B$, with $a > b$. What is the probability that $A$ stays ahead of $B$ as the votes are counted (one by one)?