ELIZABETHTOWN COLLEGE
Putnam Preparation Series
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THEME 7

1. TOPICS

Complex numbers and Geometry.

2. PRACTICE PROBLEMS

Complex Numbers

1. If a, b, and n are positive integers, prove that there exist integers x and y such that
$$(a^2+b^2)^n = x^2+y^2.$$

2. Show that
$$\arctan 1 + \arctan 2 + \arctan 3 = \pi,$$
$$4\arctan\frac{1}{5} - \arctan\frac{1}{239} = \frac{\pi}{4}.$$

3. Evaluate each of the following expressions:
$$\binom{2001}{0} - \binom{2001}{2} + \binom{2001}{4} - \binom{2001}{6} + \cdots - \binom{2001}{1998} + \binom{2001}{2000},$$
$$\binom{2002}{0} + \binom{2002}{4} + \binom{2002}{8} + \cdots + \binom{2002}{2000},$$
$$1 + \frac{1}{4!} + \frac{1}{7!} + \frac{1}{10!} + \frac{1}{13!} + \cdots.$$

4. Find constants $a_0, a_1, \ldots, a_6$ so that
$$\cos^6\theta = a_6\cos 6\theta + a_5\cos 5\theta + \cdots + a_1\cos\theta + a_0.$$

5. Express $\cos 5\theta$ in terms of $\cos\theta$.

6. Let $\cos\theta = 1/\pi$. Evaluate

$$\sum_{n=0}^{\infty} \frac{\cos(n\theta)}{2^n}.$$

7. For which ordered pairs (b, c) of real numbers do both roots of the quadratic equation $z^2 + bz + c = 0$ lie inside the unit disk $|z| < 1$ in the complex plane? Draw a reasonably accurate graph of the region in the bc-plane for which this condition holds. Identify precisely the boundary curves of this region.

8. **(1989A3)** Prove that if

$$11z^{10} + 10iz^9 + 10iz - 11 = 0,$$

then $|z| = 1$. (Here z is a complex number and $i^2 = -1$.)

9. Let $f(z) = |z^{1000} - z^5 + 1|$, where z is a complex number on the unit circle. Find, with proof, the maximum and minimum values of $f(z)$.

10. For integer $n \geq 2$ show that

$$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}\cdots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}.$$

11. For positive integer n define

$$S_n = \binom{3n}{0} + \binom{3n}{3} + \binom{3n}{6} + \cdots + \binom{3n}{3n}.$$

Find a closed-form expression for S_n, and prove that

$$\lim_{n\to\infty} \sqrt[3n]{S_n} = 2.$$

12. Let $n = 2m$ where m is an odd integer greater than 1. Let $\theta = e^{2\pi i/n}$. Find a finite list of integers $a_0, a_1, \ldots, a_k$ such that

$$\frac{1}{1-\theta} = a_k\theta^k + a_{k-1}\theta^{k-1} + \cdots + a_1\theta + a_0.$$

13. Let k be a positive integer, let $m = 2^k + 1$, and let $r \neq 1$ be a complex root of $z^m - 1 = 0$. Prove that there exist polynomials $P(z)$ and $Q(z)$ with integer coefficients such that

$$(P(r))^2 + (Q(r))^2 = -1.$$

14. **(1987A1)** Curves A, B, C, and D are defined in the plane as follows:

$$\begin{aligned} A &= \left\{(x,y) : x^2 - y^2 = \frac{x}{x^2+y^2}\right\}, \\ B &= \left\{(x,y) : 2xy + \frac{y}{x^2+y^2} = 3\right\}, \\ C &= \left\{(x,y) : x^3 - 3xy^2 + 3y = 1\right\}, \\ D &= \left\{(x,y) : 3x^2y - 3x - y^3 = 0\right\}. \end{aligned}$$

Prove that $A \cap B = C \cap D$.

15. **(1991B2)** Suppose f and g are nonconstant, differentiable, real-valued functions on $I\!R$. Furthermore, suppose that for each pair of real numbers x and y,

$$\begin{aligned} f(x+y) &= f(x)f(y) - g(x)g(y), \\ g(x+y) &= f(x)g(y) + g(x)f(y). \end{aligned}$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

16. **(1985A5)** Let

$$I_m = \int_0^{2\pi} \cos(x)\cos(2x)\cdots\cos(mx)\,dx.$$

For which integers m, $1 \leq m \leq 10$, is $I_m \neq 0$?

17. Let

$$G_n = x^n \sin(nA) + y^n \sin(nB) + z^n \sin(nC),$$

where x, y, z, A, B, C are real and $A + B + C$ is an integral multiple of π. Prove that if $G_1 = G_2 = 0$, then $G_n = 0$ for all positive integers n.

18. In a triangle ABC the points D, E, and F trisect the sides so that $BC = 3BD$, $CA = 3CE$, and $AB = 3AF$. Show that the triangles ABC and DEF have the same centroid.

19. Let C be a circle with center O, and Q a point inside C different from O. Show that the area enclosed by the locus of the centroid of triangle OPQ as P moves about the circumference of C is idependent of Q.

20. Given a point P on the circumference of a unit circle and the vertices A_1, A_2, $\ldots$, A_n of an inscribed regular polygon of n sides, prove that $|PA_1|^2 + |PA_2|^2 + \cdots + |PA_n|^2$ is constant.

21. **(2004B4)** Let n be a positive integer, $n \geq 2$, and put $\theta = 2\pi/n$. Define points $P_k = (k, 0)$ in the xy-plane, for $k = 1, 2, \ldots, n$. Let R_k be the map that rotates the plane counterclockwise by the angle θ about the point P_k. Let R denote the map obtained by applying, in order, R_1, then $R_2, \ldots$, then R_n. For an arbitrary point (x, y), find, and simplify, the coordinates of $R(x, y)$.

22. A_1, A_2, $\ldots$, A_n are vertices of a regular polygon inscribed in a circle of radius R and center O. P is a point on OA_1 extended beyond A_1. Show that
$$\prod_{k=1}^{n} |PA_k| = |OP|^n - r^n.$$

23. A regular n-sided polygon is inscribed in a unit circle. Find the product of the lengths of all its sides and diagonals.

24. Prove that if the points in the complex plane corresponding to two distinct complex numbers z_1 and z_2 are two vertices of an equilateral triangle, then the third vertex corresponds to $-\omega z_1 - \omega^2 z_2$, where ω is a non-real cube root of unity.

25. Suppose $ABCD$ is a convex plane quadrilateral. Construct a square with side AB outwards (i.e. not overlapping with the quadrilateral). Do the same with the other three sides. If L and M are the line segments joining the midpoints of opposite squares, show that L and M are perpendicular, and have the same length.

Other Problems in Geometry

26. **(1998A2)** Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A+B$ depends only on the arc length, and not on the position, of s.

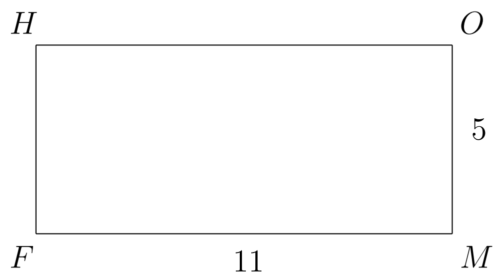
27. **(1990A3)** Prove that any convex pentagon whose vertices (no three of which are collinear) have integer coordinates must have area $\geq 5/2$.

28. **(2004A2)** For $i = 1, 2$ let T_i be a triangle with side lengths a_i, b_i, c_i, and area A_i. Suppose that $a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2$, and that T_2 is an acute triangle. Does it follow that $A_1 \leq A_2$?

29. **(1994A3)** Show that if the points of an isosceles right triangle of side length 1 are each colored with one of four colors, then there must be two points of the same color which are at least a distance $2 - \sqrt{2}$ apart.

30. **(1996A2)** Let C_1 and C_2 be circles whose centers are 10 units apart, and whose radii are 1 and 3. Find, with proof, the locus of all points M for which there exists points X on C_1 and Y on C_2 such that M is the midpoint of the line segment XY.

31. **(1997A1)** A rectangle, $HOMF$, has sides $HO = 11$ and $OM = 5$. A triangle ABC has H as the intersection of the altitudes, O the center of the circumscribed circle, M the midpoint of BC, and F the foot of the altitude from A. What is the length of BC?



32. **(2000A5)** Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.

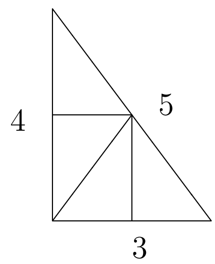
33. **(1998A6)** Let A, B, C denote distinct points with integer coordinates in $\mathbb{R}^2$. Prove that if

$$(|AB| + |BC|)^2 < 8 \cdot [ABC] + 1$$

then A, B, C are three vertices of a square. Here $|XY|$ is the length of segment XY and $[ABC]$ is the area of triangle ABC.

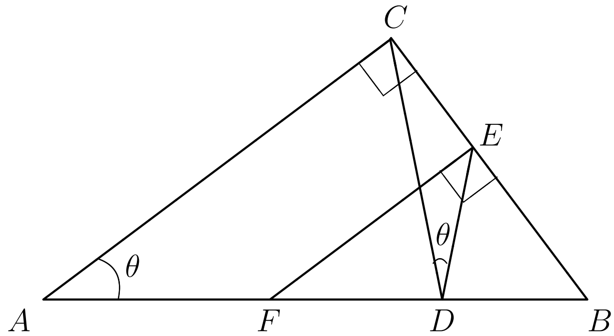
34. **(2000A3)** The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5, and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.

35. **(1997B6)** The dissection of the 3–4–5 triangle shown below has diameter $5/2$.



Find the least diameter of a dissection of this triangle into four parts. (The diameter of a dissection is the least upper bound of the distances between pairs of points belonging to the same part.)

36. **(1999B1)** Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta \to 0} |EF|$. [Here $|PQ|$ denotes the length of the line segment PQ.]



37. **(1989B5)** Label the vertices of a trapezoid T (quadrilateral with two parallel sides) inscribed in the unit circle as A, B, C, D so that AB is parallel to CD and A, B, C, D are in counterclockwise order. Let s_1, s_2, and d denote the lengths of the line segments AB, CD, and OE, where E is the point of intersection of the diagonals of T, and O is the center of the circle. Determine the least upper bound of $(s_1 - s_2)/d$ over all such T for which $d \neq 0$, and describe all cases, if any, in which it is attained.

38. **(2001A4)** Triangle ABC has an area 1. Points E, F, G lie, respectively, on sides BC, CA, AB such that AE bisects BF at point R, BF bisects CG at point S, and CG bisects AE at point T. Find the area of the triangle RST.