ELIZABETHTOWN COLLEGE
Putnam Preparation Series
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THEME 8

1. TOPICS

Inequalities show up very often in different forms. Some of them are very tricky, but most can be derived from simple principles.

Squares are nonnegative. The trivial inequality $x^2 \geq 0$ can be very useful. We can derive from it, for positive a, b, c, x, y,

$$\frac{x+y}{2} \geq \sqrt{xy},$$

or

$$a^2 + b^2 + c^2 \geq ab + bc + ca,$$

etc.

Convexity. Let f be a real-valued function defined on an interval I. The function f is called convex if for every choice of $x, y \in I$ and $\mu \in [0, 1]$,

$$f(\mu x + (1-\mu)y) \leq \mu f(x) + (1-\mu)f(y).$$

This can be generalized to a greater number of points. If $x_1, x_2, \ldots, x_n \in I$ and $t_1, t_2, \ldots, t_n$ are non-negative numbers such that $t_1 + t_2 + \cdots + t_n = 1$, then

$$f(t_1x_1 + t_2x_2 + \cdots + t_nx_n) \leq t_1f(x_1) + t_2f(x_2) + \cdots + t_nf(x_n).$$

Passing to a limit, we obtain the integral version

$$f\left(\int_I x\,p(x)\,dx\right) \leq \int_I f(x)p(x)\,dx,$$

where $p(x) \geq 0$ for $x \in I$ and $\int_I p(x)\,dx = 1$.

The last two equations are particular cases of the Jensen's inequality:

$$f(\mathbf{E}X) \leq \mathbf{E}f(X).$$

Looking at endpoints. If a function is linear or, more generally, convex, it attains its maximum at an endpoint. Using this simple observation, we can show, for example, that $0 \leq x, y, z \leq 1$ implies

$$1 + xy + yz + zx \geq x + y + z \geq xy + yz + zx.$$

Aritmetic, geometric, harmonic, and quadratic means. Let $x_1, x_2, \ldots, x_n$ be positive numbers. Define

$$\begin{aligned} A.M. &= \frac{x_1 + x_2 + \cdots + x_n}{n} \\ G.M. &= \sqrt[n]{x_1 x_2 \cdots x_n} \\ H.M. &= \frac{n}{x_1^{-1} + x_2^{-1} + \cdots + x_n^{-1}} \\ Q.M. &= \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}} \end{aligned}$$

We have

$$H.M. \le G.M. \le A.M. \le Q.M.$$

In fact, all of the above inequalities are strict, unless $x_1 = x_2 = \cdots = x_n$.

More generally, it follows from Jensen's inequality that, if $p < q$, then

$$\left(\frac{x_1^p + x_2^p + \cdots + x_n^p}{n} \right)^{1/p} \le \left(\frac{x_1^q + x_2^q + \cdots + x_n^q}{n} \right)^{1/q}$$

with equality only when $x_1 = x_2 = \cdots = x_n$.

The $A.M.$-$H.M.$ inequality can be rewritten into the following "product" form:

$$(x_1 + x_2 + \cdots + x_n) \left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \right) \ge n^2.$$

The Cauchy-Schwartz inequality. Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be real numbers. Then

$$\left| \sum_{k=1}^n x_k y_k \right| \le \left(\sum_{k=1}^n x_k \right)^{1/2} \left(\sum_{k=1}^n y_k \right)^{1/2},$$

with equality only if the two vectors $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are proportional to each other.

2. PRACTICE PROBLEMS

1. Let $x_1, x_2, \ldots, x_n$ be real numbers. Show that

$$x_1^2 + x_2^2 + \cdots + x_n^2 \geq \frac{2}{n-1} \sum_{1 \leq i < j \leq n} x_i x_j.$$

2. For positive a, b, c prove that

$$b^3c^3 + c^3a^3 + a^3b^3 \geq 3a^2b^2c^2.$$

3. For positive $x_1, x_2, \ldots, x_n$ prove that

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} \geq n.$$

4. If $x \leq y \leq z$ and $y > 0$ prove that

$$x + z - y \geq \frac{xz}{y}.$$

5. For non-negative $u_1, u_2, \ldots, u_n$ prove that

$$\left(\sum_{i=1}^{n} u_i \right)^3 \leq n^2 \sum_{i=1}^{n} u_i^3.$$

6. Let

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}.$$

   Prove that

$$n(\sqrt[n]{n+1} - 1) \leq H_n \leq n - \frac{n-1}{\sqrt[n-1]{n}}.$$

7. If a, b, c are positive, show that

$$a^a b^b c^c \geq (abc)^{(a+b+c)/3}.$$

8. **(2003A2)** Let a_1, a_2, $\ldots$, a_n, and b_1, b_2, $\ldots$, b_n, be non-negative real numbers. Show that

$$(a_1 a_2 \ldots a_n)^{1/n} + (b_1 b_2 \ldots b_n)^{1/n} \leq ((a_1 + b_1)(a_2 + b_2) \ldots (a_n + b_n))^{1/n}.$$

9. Given n points on the unit sphere $x^2+y^2+z^2=1$, prove that the sum of the squares of distances between them is at most n^2.

10. Prove that
$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{999999}{1000000}<\frac{1}{1000}.$$

11. Suppose x_1, x_2, ..., x_n are positive real numbers. Prove that
$$\frac{x_1}{x_1+x_2}+\frac{x_2}{x_2+x_3}+\cdots+\frac{x_{n-1}}{x_{n-1}+x_n}+\frac{x_n}{x_n+x_1}\geq 1.$$

12. Suppose x_1, x_2, ..., x_n are positive real numbers. Prove that
$$\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_4}+\cdots+\frac{x_{n-1}}{x_n+x_1}+\frac{x_n}{x_1+x_2}\geq\frac{n}{4}.$$

13. Prove or disprove: If x and y are real numbers with $y\geq 0$ and $y(y+1)\leq(x+1)^2$, then $y(y-1)\leq x^2$.

14. Let a, b, c, be positive real numbers, such that $abc=1$. Prove that
$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}.$$

15. **(2002B3)** Show that, for all integers $n>1$,
$$\frac{1}{2ne}<\frac{1}{e}-\left(1-\frac{1}{n}\right)^n<\frac{1}{ne}.$$

16. **(1991A5)** Find the maximum value of
$$\int_0^y\sqrt{x^4+(y-y^2)^2}\,dx$$
for $0\leq y\leq 1$.

17. **(1991B6)** Let a and b be positive numbers. Find the largest number c, in terms of a and b, such that
$$a^xb^{1-x}\leq a\frac{\sinh ux}{\sinh u}+b\frac{\sinh u(1-x)}{\sinh u}$$
for all u with $0<|u|\leq c$ and for all x, $0<x<1$. (Note: $\sinh u=(e^u-e^{-u})/2$.)

18. **(1996B3)** Given that $\{x_1, x_2, \ldots, x_n\} = \{1, 2, \ldots, n\}$, find, with proof, the largest possible value, as a function of n (with $n \geq 2$), of

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1.$$

19. **(1998A3)** Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$

20. **(1999B4)** Let f be a real function with a continuous third derivative such that $f(x), f'(x), f''(x), f'''(x)$ are positive for all x. Suppose that $f'''(x) \leq f(x)$ for all x. Show that $f'(x) < 2f(x)$ for all x.

21. Let n be a natural number, and let $x_k \in [0, 1]$ for $k = 1, 2, \ldots, n$. Find the maximum of the sum

$$\sum_{k<j} |x_k - x_j|.$$

22. **(1978A5)** Let $0 < x_i < \pi$ for $i = 1, 2, \ldots, n$ and set

$$x = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

Prove that

$$\prod_{i=1}^{n} \frac{\sin x_i}{x_i} \leq \left(\frac{\sin x}{x}\right)^n.$$

23. **(1979B6)** For $k = 1, 2, \ldots, n$ let $z_k = x_k + iy_k$, where x_k and y_k are real and $i = \sqrt{-1}$. Let r be the absolute value of the real part of

$$\pm\sqrt{z_1^2 + z_2^2 + \cdots + z_n^2}.$$

Prove that $r \leq |x_1| + |x_2| + \cdots + |x_n|$.

24. **(1982B6)** Let $K(x, y, z)$ denote the area of a triangle whose sides have lengths x, y, and z. For any two triangles with sides a, b and c, and a', b', and c' respectively, prove that

$$\sqrt{K(a, b, c)} + \sqrt{K(a', b', c)'} \leq \sqrt{K(a + a', b + b', c + c')}$$

and determine the cases of equality.

25. **(2004A6)** Suppose that $f(x,y)$ is a continuous real-valued function on the unit square $0 \le x \le 1, 0 \le y \le 1$. Show that

$$\int_0^1 \left(\int_0^1 f(x,y) dx \right)^2 dy + \int_0^1 \left(\int_0^1 f(x,y) dy \right)^2 dx$$
$$\le \left(\int_0^1 \int_0^1 f(x,y) dx\, dy \right)^2 + \int_0^1 \int_0^1 (f(x,y))^2\, dx\, dy.$$