ELIZABETHTOWN COLLEGE
Putnam Preparation Series
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THEME 10

1. TOPICS

Today we will look at various problems from Real Analysis and Calculus.

2. PRACTICE PROBLEMS

Real Analysis

1. **(1999A5)** Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then

$$|p(0)| \leq C \int_{-1}^{1} |p(x)|\, dx.$$

2. **(1990B5)** Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for $n = 1, 2, 3, \ldots$ the polynomial

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

has exactly n distinct real roots?

3. **(1989B4)** Can a countably infinite set have an uncountable collection of nonempty subsets such that the intersection of any two of them is finite?

4. **(1990A4)** Consider a paper punch that can be centered at any point of the plane and that, when operated, removes from the plane precisely those points whose distance from the center is irrational. How many punches are needed to remove every point?

5. **(1990A2)** Is $\sqrt{2}$ the limit of a sequence of numbers of the form

$$\sqrt[3]{n} - \sqrt[3]{m}$$

$(n, m = 0, 1, 2, \ldots)$?

6. **(1998A5)** Let $\mathcal{F}$ be a finite collection of open discs in $\mathbb{R}^2$ whose union contains a set $E \subseteq \mathbb{R}^2$. Show that there is a pairwise disjoint subcollection $D_1, \ldots, D_n$ in $\mathcal{F}$ such that

$$E \subseteq \cup_{j=1}^{n} 3D_j.$$

Here, if D is the disc of radius r and center P, then $3D$ is the disc of radius $3r$ and center P.

7. **(1995B6)** For a positive real number α, define

$$S(\alpha) = \{\lfloor n\alpha \rfloor : n = 1, 2, 3, \ldots\}.$$

Prove that $\{1, 2, 3, \ldots\}$ cannot be expressed as the disjoint union of three sets $S(\alpha), S(\beta)$ and $S(\gamma)$.

8. **(1994A5)** Let $(r_n)_{n \geq 0}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} r_n = 0$. Let S be the set of numbers representable as a sum

$$r_{i_1} + r_{i_2} + \cdots + r_{i_{1994}}$$

with $i_1 < i_2 < \cdots < i_{1994}$. Show that every nonempty interval (a, b) contains a nonempty subinterval (c, d) that does not intersect S.

9. **(1986B4)** For a positive real number r, let $G(r)$ be the minimum value of $|r - \sqrt{m^2 + 2n^2}|$ for all integers m and n. Prove or disprove the assertion that $\lim_{r\to\infty} G(r)$ exists and equals 0.

10. **(1992A4)** Let f be an infinitely differentiable real-valued function defined on the real numbers. If

$$f\left(\frac{1}{n}\right) = \frac{n^2}{n^2+1}, \qquad n = 1, 2, 3, \ldots,$$

compute the values of the derivatives $f^{(k)}(0)$, $k = 1, 2, 3, \ldots$.

11. Compute the limit:

$$\lim_{x\to 0} \frac{\sin\tan x - \tan\sin x}{\arcsin\arctan x - \arctan\arcsin x}.$$

Calculus

12. Is it possible to represent the function $\sin x$ as the difference of two convex functions?

13. **(2006A1)** Find the volume of the region of points (x, y, z) such that

$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).$$

14. **(1989A2)** Evaluate $\displaystyle\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}}\, dy\, dx$ where a and b are positive.

15. **(1985B5)** Evaluate $\int_0^\infty t^{-1/2} e^{-1985(t+t^{-1})}\, dt$. You may assume that $\int_{-\infty}^\infty e^{-x^2}\, dx = \sqrt{\pi}$.

16. **(2000A4)** Show that the improper integral

$$\lim_{B\to\infty} \int_0^B \sin(x)\sin(x^2)\, dx$$

converges.

17. **(1995A2)** For what pairs (a, b) of positive real numbers does the improper integral

$$\int_b^\infty \left(\sqrt{\sqrt{x+a} - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x-b}} \right) dx$$

converge?

18. **(2001A6)** Can an arc of a parabola inside a circle of radius 1 have a length greater than 4?

19. **(1998B3)** Let H be the unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$, C the unit circle $\{(x, y, 0) : x^2 + y^2 = 1\}$, and P the regular pentagon inscribed in C. Determine the surface area of that portion of H lying over the planar region inside P, and write your answer in the form $A \sin\alpha + B\cos\beta$, where A, B, α, β are real numbers.

20. **(1991A4)** Does there exist an infinite sequence of closed discs D_1, D_2, $D_3, \ldots$ in the plane, with centers c_1, c_2, $c_3, \ldots$, respectively, such that

    (i) the c_i have no limit point in the finite plane,

    (ii) the sum of the areas of the D_i is finite, and

    (iii) every line in the plane intersects at least one of the D_i?

21. **(1996B2)** Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

22. **(1994B2)** For which real numbers c is there a straight line that intersects the curve

$$y = x^4 + 9x^3 + cx^2 + 9x + 4$$

in four distinct points?

23. **(1984A5)** Let R be the region consisting of all triples (x, y, z) of nonnegative real numbers satisfying $x + y + z \leq 1$. Let $w = 1 - x - y - z$. Express the value of the integral

$$\int\!\!\int\!\!\int_R x^1 y^9 z^8 w^4 \, dx \, dy \, dz$$

in the form $a!b!c!d!/n!$ where a, b, c, d, and n are positive integers.

24. **(1994B3)** Find the set of all real numbers k with the following property: For any positive, differentiable function f that satisfies $f'(x) > f(x)$ for all x, there is some number N such that $f(x) > e^{kx}$ for all $x > N$.

25. **(2004B3)** Determine all real numbers $a > 0$ for which there exists a nonnegative continuous function $f(x)$ defined on $[0, a]$ with the property that the region

$$R = \{(x, y); 0 \leq x \leq a, 0 \leq y \leq f(x)\}$$

has perimeter k units and area k square units for some real number k.

26. **(2002A6)** Fix an integer $b \geq 2$. Let $f(1) = 1$, $f(2) = 2$, and for each $n \geq 3$, define $f(n) = nf(d)$, where d is the number of base-b digits of n. For which values of b does

$$\sum_{n=1}^{\infty} \frac{1}{f(n)}$$

converge?

27. **(2004B5)** Evaluate

$$\lim_{x \to 1^-} \prod_{n=0}^{\infty} \left(\frac{1 + x^{n+1}}{1 + x^n} \right)^{x^n}.$$

28. Compute the integrals:

(**a**) $\displaystyle\int_0^\infty \frac{2\sin x - \sin 2x}{x^2} dx$ (**b**) $\displaystyle\int_0^\infty \frac{\arctan \pi x - \arctan x}{x} dx.$

(**c**) $\displaystyle\int_0^1 \frac{\arcsin x^3 - \arcsin x^2}{x \ln x} dx$

29. **(1987A5)** Let

$$\vec{G}(x, y) = \left(\frac{-y}{x^2 + 4y^2}, \frac{x}{x^2 + 4y^2}, 0 \right).$$

Prove or disprove that there is a vector-valued function

$$\vec{F}(x, y, z) = (M(x, y, z), N(x, y, z), P(x, y, z))$$

with the following properties:

(i) M, N, P have continuous partial derivatives for all $(x, y, z) \neq (0, 0, 0)$;

(ii) Curl $\vec{F} = \vec{0}$ for all $(x, y, z) \neq (0, 0, 0)$;

(iii) $\vec{F}(x, y, 0) = \vec{G}(x, y)$.

30. **(1986A5)** Suppose $f_1(x), f_2(x), \ldots, f_n(x)$ are functions of n real variables $x = (x_1, \ldots, x_n)$ with continuous second-order partial derivatives everywhere on $\mathbb{R}^n$. Suppose further that there are constants c_{ij} such that
$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = c_{ij}$$
for all i and j, $1 \le i \le n$, $1 \le j \le n$. Prove that there is a function $g(x)$ on $\mathbb{R}^n$ such that $f_i + \partial g/\partial x_i$ is linear for all i, $1 \le i \le n$. (A linear function is one of the form
$$a_0 + a_1x_1 + a_2x_2 + \cdots + a_nx_n.)$$

31. **(2003A3)** Find the minimum value of
$$|\sin x + \cos x + \tan x + \cot x + \sec x + \csc x|$$
for real numbers x.

Differential Equations

32. **(1988A2)** A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).

33. **(1995A5)** Let $x_1, x_2, \ldots, x_n$ be differentiable (real-valued) functions of a single variable t which satisfy
$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
\frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n
\end{aligned}$$
for some constants $a_{ij} > 0$. Suppose that for all i, $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_1, x_2, \ldots, x_n$ necessarily linearly dependent?

34. **(1989B3)** Let f be a function on $[0,\infty)$, differentiable and satisfying

$$f'(x) = -3f(x) + 6f(2x)$$

for $x > 0$. Assume that $|f(x)| \leq e^{-\sqrt{x}}$ for $x \geq 0$ (so that $f(x)$ tends rapidly to 0 as x increases). For n a nonnegative integer, define

$$\mu_n = \int_0^\infty x^n f(x)\, dx$$

(sometimes called the nth moment of f).

a. Express μ_n in terms of μ_0.

b. Prove that the sequence $\{\mu_n 3^n/n!\}$ always converges, and that the limit is 0 only if $\mu_0 = 0$.

35. **(1997B2)** Let f be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x),$$

where $g(x) \geq 0$ for all real x. Prove that $|f(x)|$ is bounded.

36. **(1987A3)** For all real x, the real-valued function $y = f(x)$ satisfies

$$y'' - 2y' + y = 2e^x.$$

(a) If $f(x) > 0$ for all real x, must $f'(x) > 0$ for all real x? Explain.

(b) If $f'(x) > 0$ for all real x, must $f(x) > 0$ for all real x? Explain.

37. **(1990B1)** Find all real-valued continuously differentiable functions f on the real line such that for all x,

$$(f(x))^2 = \int_0^x [(f(t))^2 + (f'(t))^2]\, dt + 1990.$$

38. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(x+y) = f(x) + f(y)$. Show that $f(x) = cx$ for some $c \in \mathbb{R}$.

39. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function satisfying:

    (a) $f(x, y) = f(y, x)$

    (b) $f(x, x) = x$

    (c) if $y > x$, then $f(x, y) = f(x, y - x)$.

    What is the function f?

40. **(IMO1968)** Let $f : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation

$$f(x+a) = 1/2 + \sqrt{f(x) - f(x)^2}$$

    for some fixed $a > 0$. Prove that f is periodic, and give an example of a nonconstant f satisfying the equation for $a = 1$.

41. **(1992A1)** Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions:

    (i) $f(f(n)) = n$, for all integers n;

    (ii) $f(f(n+2)+2) = n$ for all integers n;

    (iii) $f(0) = 1$.

42. **(2000B4)** Let $f(x)$ be a continuous function such that $f(2x^2 - 1) = 2xf(x)$ for all x. Show that $f(x) = 0$ for $-1 \leq x \leq 1$.

43. **(1996A6)** Let $c \geq 0$ be a constant. Give a complete description, with proof, of the set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = f(x^2 + c)$ for all $x \in \mathbb{R}$.

44. **(1988A5)** Prove that there exists a *unique* function f from the set $\mathbb{R}^+$ of positive real numbers to $\mathbb{R}^+$ such that

$$f(f(x)) = 6x - f(x) \quad \text{and} \quad f(x) > 0 \quad \text{for all } x > 0.$$

45. **(2001B5)** Let a and b be real numbers in the interval $(0, 1/2)$, and let g be a continuous real-valued function such that $g(g(x)) = ag(x) + bx$ for all real x. Prove that $g(x) = cx$ for some constant c.

46. **(1993B4)** The function $K(x, y)$ is positive and continuous for $0 \leq x \leq 1, 0 \leq y \leq 1$, and the functions $f(x)$ and $g(x)$ are positive and continuous for $0 \leq x \leq 1$. Suppose that for all x, $0 \leq x \leq 1$,

$$\int_0^1 f(y)K(x, y)\, dy = g(x) \quad \text{and} \quad \int_0^1 g(y)K(x, y)\, dy = f(x).$$

Show that $f(x) = g(x)$ for $0 \leq x \leq 1$.

47. **(1991B2)** Suppose f and g are non-constant, differentiable, real-valued functions on $I\!R$. Furthermore, suppose that for each pair of real numbers x and y,

$$\begin{aligned} f(x + y) &= f(x)f(y) - g(x)g(y), \\ g(x + y) &= f(x)g(y) + g(x)f(y). \end{aligned}$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

48. **(JapMO2004)** Find all functions $f : I\!R \to I\!R$ such that for every two real numbers x and y:

$$f\left(x\, f(x) + f(y)\right) = f(x)^2 + y.$$