ELIZABETHTOWN COLLEGE
Putnam Preparation Series
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THEME 11

1. TOPICS

Today we will look at some two-person (and other) games.

2. PRACTICE PROBLEMS

Matchstick Games

Explain who can win the following games by playing perfectly, and how. Test your arguments by playing against other people. The matchstick games (also called sometimes Nim-type games) work as follows. Several piles of matches (or stones, or checkers, or beans) are given. Two players alternate playing. Each play ("move") involves removing a certain number of matches from one (or more) of the piles. The last person to make a move wins. Also, the misère version of the games can be considered: the last person to play loses.

1. *Bachet's Game.* Each player can remove between one and k matches. The initial pile has n matches. It is assumed that $n > k > 1$.

2. Each player can remove 2^m matches, for any non-negative integer m.

3. There are four piles of matches, with 7, 8, 9, and 10 matches respectively. Each player can remove between one and three matches from one of the piles.

4. There is one pile of matches. When the pile has n matches left, the next player may remove up to $2\sqrt{n+1} - 2$ matches. (The game ends when there is one match left.)

5. There are four piles of matches. If there are n matches in a pile, then the next player may remove 2^m matches from that pile, where 2^m appears in the binary representation of n.

6. *Wythoff's Game.* There are two piles of matches on the table. A player's move consists in either removing any number of matches from one of the piles, or removing the same number of matches from both. The winner is the one to take the last match.

7. **(1995B5)** A game starts with four heaps of beans, containing 3, 4, 5 and 6 beans. The two players move alternately. A move consists of taking **either**

   (a) one bean from a heap, provided at least two beans are left behind in that heap, **or**

   (b) a complete heap of two or three beans.

   The player who takes the last heap wins. To win the game, do you want to move first or second? Give a winning strategy.

Other Games

8. Show that there is an optimal strategy for chess.

9. Two players alternately replace the stars in

$$x^4 + \star x^3 + \star x^2 + \star x + \star = 0$$

   by integers of their choice. The first player wins if he gets a polynomial without integer roots after the fourth step. Otherwise, the second player wins. If playing optimally, who wins and how?

10. The game of Toe-Tac-Tic is played like the usual Tic-Tac-Toe, except that the first player to get three in a row *loses.*

    (a) Show that, if the first player starts in the center, then he can force a draw.

    (b) Harder: Show that if the first player plays his first move at any of the eight non-central cells, then the second player can win.

11. **(2002A4)** In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

12. *The fifteen game.* (This one is more a trick than a problem.) Nine cards are on the table, face up, numbered one through nine. The two players alternate picking up cards.The first player to have three cards summing to fifteen wins. If all cards are picked up without either player winning, the game is declared a draw. Show that

    (a) if both players play perfectly, the game will be drawn, and

    (b) if one player knows what's going on, she can do very well, for example by starting with any of the even cards.

    (Hint: There is a really magic way of reducing this game to another one, a very famous, too).

13. **(2002B4)** An integer n, unknown to you, has been randomly chosen in the interval $[1, 2002]$ with uniform probability. Your objective is to select n in an **odd** number of guesses. After each incorrect guess, you are informed whether n is higher or lower, and you **must** guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than $2/3$.

14. **(2006A2)** Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

15. **(1993B2)** Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players, A and B. Beginning with A, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n + 1$. The last person to discard wins the game. Assuming optimal strategy by both A and B, what is the probability that A wins?

16. *The Hats Problem.*

Below is a description of the problem, taken as an excerpt from Sara Robinson's very well written and informative article *Why Mathematicians Now Care About Their Hat Color*, New York Times, April 10, 2001. (Sara Robinson is a talented freelance journalist based in Berkeley, CA, specializing in mathematics, computer science, and economics. Look her up online, she has many interesting articles.)

"Three players enter a room and a red or blue hat is placed on each person's head. The color of each hat is determined by a coin toss, with the outcome of one coin toss having no effect on the others. Each person can see the other players' hats but not his own. No communication of any sort is allowed, except for an initial strategy session before the game begins. Once they have had a chance to look at the other hats, the players must simultaneously guess the color of their own hats or pass. The group shares a hypothetical $3 million prize if at least one player guesses correctly and no players guess incorrectly. The same game can be played with any number of players. The general problem is to find a strategy for the group that maximizes its chances of winning the prize. One obvious strategy for the players, for instance, would be for one player to always guess "red" while the other players pass. This would give the group a 50 percent chance of winning the prize. Can the group do better? Most mathematicians initially think not. Since each person's hat color is independent of the other players' colors and no communication is allowed, it seems impossible for the players to learn anything just by looking at one another. All the players can do, it seems, is guess."

(a) Show a strategy in the three-player game which lets the team win with probability $3/4$.

(b) What can you say about the seven-player game?