Dept. of Math. Sci., WPI
MA 1034 Analysis 4
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Homework Assignment 1

Solutions

1. Find an equation of a sphere that has a center at the point $(5, 3, -6)$ and touches the yz-plane.

   SOLUTION. The radius connecting the center of the sphere and the point where it touches the yz-plane is perpendicular to the yz-plane. In other words, the point A where the sphere touches the yz-plane is the orthogonal projection of the center $C(5, 3, -6)$ onto the yz-plane. This means that A has coordinates $(0, 3, -6)$. The radius is $r = \|\overrightarrow{CA}\| = 5$. The equation of the sphere is then

   $$(x - 5)^2 + (y - 3)^2 + (z + 6)^2 = 25.$$

2. Find the equation of the sphere with a diameter determined by its endpoints, $(1, -2, 7)$ and $(9, 0, 1)$.

   SOLUTION. The length of the diameter is

   $$\sqrt{(1-9)^2 + (-2-0)^2 + (7-1)^2} = \sqrt{104} = 2\sqrt{26}.$$

   The radius is then $r = \sqrt{26}$. The center is the midpoint of the line segment with endpoints $(1, -2, 7)$ and $(9, 0, 1)$, i.e. the center has coordinates

   $$\left(\frac{1+9}{2}, \frac{-2+0}{2}, \frac{7+1}{2}\right) = (5, -1, 4).$$

   Putting this information together, we get the equation

   $$(x - 5)^2 + (y + 1)^2 + (z - 4)^2 = 26.$$

3. Find the center and the radius of a sphere with an equation

   $$2x^2 + 2y^2 + 2z^2 + 4y - 2z = 1.$$

   SOLUTION. We rearrange the terms and complete the squares:

   $$\begin{aligned} 2x^2 + 2y^2 + 2z^2 + 4y - 2z &= 1 \\ x^2 + y^2 + z^2 + 2y - z &= \frac{1}{2} \\ x^2 + (y^2 + 2y) + (z^2 - z) &= \frac{1}{2} \\ x^2 + (y+1)^2 + (z - \frac{1}{2})^2 &= \frac{1}{2} + 1^2 + (\frac{1}{2})^2 \\ x^2 + (y+1)^2 + (z - \frac{1}{2})^2 &= \frac{7}{4} \end{aligned}$$

   Which gives us the center $(0, -1, \frac{1}{2})$ and radius $\frac{\sqrt{7}}{2}$

4. Two spheres with equations

$$x^2 + y^2 + z^2 - 4x - 2y - 6z + 12 = 0$$

and

$$x^2 + y^2 + z^2 - 2x + 2y - 10z + 24 = 0$$

respectively, intersect. Their intersection is a circle in $I\!R^3$. Find the center and the radius of this cirle.

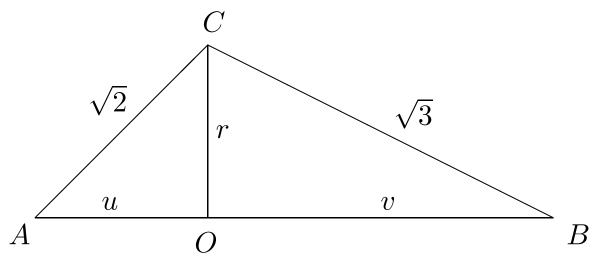
Solution. As in the previous problem, we rewrite the equations in a form which gives us the centers and radii:

$$(x-2)^2 + (y-1)^2 + (z-3)^2 = 2$$

and

$$(x-1)^2 + (y+1)^2 + (z-5)^2 = 3.$$

The first sphere has a center $A(2,1,3)$ and radius $\sqrt{2}$. The second sphere has a center $B(1,-1,5)$ and radius $\sqrt{3}$. Let C be a point on the intersection of the two spheres. Then $|AB| = \sqrt{1^2+2^2+2^2} = 3$, $|AC| = \sqrt{2}$, and $|BC| = \sqrt{3}$. Since three points always lie in a plane, we can put A, B, and C on a planar picture as follows:



Let O be the orthogonal projection of C onto AB. Then O is the center of the circle and $|OC|$ is its radius. Denoting $|AO| = u$, $|OB| = v$, $|OC| = r$, we get

$$\begin{aligned} u + v &= 3 \\ u^2 + r^2 &= 2 \\ v^2 + r^2 &= 3 \end{aligned}$$

Solving this system, we get $r = \sqrt{2}/3$, $u = 4/3$, and $v = 5/3$. Thus, the radius of the circle is $r = \sqrt{2}/3$, and the center is the point O which divides the line segment AB into a ratio $4:5$. Therefore the center O has coordinates

$$\frac{5}{9}(2,1,3) + \frac{4}{9}(1,-1,5) = \left(\frac{14}{9}, \frac{1}{9}, \frac{35}{9}\right).$$

5. Write the vector $\vec{u} = \langle 1, 0, -2 \rangle$ as a linear combination of the three vectors $\langle 1,1,0 \rangle$, $\langle -1,1,-2 \rangle$, and $\langle 0,1,-2 \rangle$.

Solution. We are looking for real numbers λ_1, λ_2, λ_3, such that

$$\langle 1,0,-2 \rangle = \lambda_1 \langle 1,1,0 \rangle + \lambda_2 \langle -1,1,-2 \rangle + \lambda_3 \langle 0,1,-2 \rangle,$$

or, in components,

$$\begin{aligned} \lambda_1 - \lambda_2 &= 1 \\ \lambda_1 + \lambda_2 + \lambda_3 &= 0 \\ -2\lambda_2 - 2\lambda_3 &= -2 \end{aligned}$$

Solving this linear system, we get $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = 3$, so

$$\vec{u} = \langle 1, 0, -2 \rangle = -\langle 1, 1, 0 \rangle - 2\langle -1, 1, -2 \rangle + 3\langle 0, 1, -2 \rangle.$$

6. Let $\|\vec{a} + \vec{b}\| = \|\vec{a} - \vec{b}\|$. Find the angle between $\vec{a}$ and $\vec{b}$.

   Solution. We have:

$$\begin{aligned} \|\vec{a} + \vec{b}\| &= \|\vec{a} - \vec{b}\| \\ \|\vec{a} + \vec{b}\|^2 &= \|\vec{a} - \vec{b}\|^2 \\ (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) &= (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) \\ \|\vec{a}\|^2 + \|\vec{b}\|^2 + 2\vec{a} \cdot \vec{b} &= \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\vec{a} \cdot \vec{b} \\ 4\vec{a} \cdot \vec{b} &= 0 \\ \vec{a} \cdot \vec{b} &= 0, \end{aligned}$$

   i.e., $\vec{a}$ and $\vec{b}$ are perpendicular.

7. The Paralleloram Law states that

$$\|\vec{a} + \vec{b}\|^2 + \|\vec{a} - \vec{b}\|^2 = 2\|\vec{a}\|^2 + 2\|\vec{b}\|^2.$$

   Give a geometric interpretation of this law. Prove it.

   Solution. Let $ABCD$ be a parallelogram and denote $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{AD} = \vec{b}$. Then $\overrightarrow{DC} = \vec{a}$, $\overrightarrow{BC} = \vec{b}$, $\overrightarrow{AC} = \vec{a} + \vec{b}$, $\overrightarrow{DB} = \vec{a} - \vec{b}$. The parallelogram law states that

$$|AC|^2 + |BD|^2 = |AB|^2 + |BC|^2 + |CD|^2 + |AD|^2,$$

   i.e. the sum of the squares of the diagonals in a parallelogram is equal to the sum of the squares of the sides.

   To prove the equality, observe that

$$\begin{aligned} \|\vec{a} + \vec{b}\|^2 + \|\vec{a} - \vec{b}\|^2 &= (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) + (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) \\ &= \|\vec{a}\|^2 + \|\vec{b}\|^2 + 2\vec{a} \cdot \vec{b} + \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\vec{a} \cdot \vec{b} \\ &= 2\|\vec{a}\|^2 + 2\|\vec{b}\|^2. \end{aligned}$$

8. If $\|\vec{a}\| = \|\vec{a} - \vec{b}\|$, show that $\vec{a} \cdot \vec{b} = (\vec{b} - \vec{a}) \cdot \vec{b}$.

   Solution.

$$\begin{aligned} \|\vec{a}\| &= \|\vec{a} - \vec{b}\| \\ \|\vec{a}\|^2 &= \|\vec{a} - \vec{b}\|^2 \\ \|\vec{a}\|^2 &= (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) \\ \|\vec{a}\|^2 &= \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\vec{a} \cdot \vec{b} \\ 0 &= \vec{b} \cdot \vec{b} - 2\vec{a} \cdot \vec{b} \\ \vec{a} \cdot \vec{b} &= \vec{b} \cdot \vec{b} - \vec{a} \cdot \vec{b} \\ \vec{a} \cdot \vec{b} &= (\vec{b} - \vec{a}) \cdot \vec{b} \end{aligned}$$

9. If $\vec{c} = \|\vec{a}\|\,\vec{b} + \|\vec{b}\|\,\vec{a}$, where $\vec{a}$, $\vec{b}$, and $\vec{c}$ are all nonzero vectors, show that $\vec{c}$ bisects the angle between $\vec{a}$ and $\vec{b}$.

   SOLUTION. Since $\vec{c}$ is a linear combination ov $\vec{a}$ and $\vec{b}$, it lies in the same plane as $\vec{a}$ and $\vec{b}$. It suffices to show that the angle between $\vec{a}$ and $\vec{c}$ is the same as the angle between $\vec{b}$ and $\vec{c}$.

   The cosine of the angle between $\vec{a}$ and $\vec{c}$ is

$$\frac{\vec{a}\cdot\vec{c}}{\|\vec{a}\|\,\|\vec{c}\|} = \frac{\|\vec{a}\|(\vec{a}\cdot\vec{b}) + \|\vec{b}\|\|\vec{a}\|^2}{\|\vec{a}\|\,\|\vec{c}\|} = \frac{\vec{a}\cdot\vec{b} + \|\vec{b}\|\|\vec{a}\|}{\|\vec{c}\|},$$

   and the cosine of the angle between $\vec{b}$ and $\vec{c}$ is

$$\frac{\vec{b}\cdot\vec{c}}{\|\vec{b}\|\,\|\vec{c}\|} = \frac{\|\vec{a}\|\|\vec{b}\|^2 + \|\vec{b}\|(\vec{a}\cdot\vec{b})}{\|\vec{b}\|\,\|\vec{c}\|} = \frac{\|\vec{b}\|\|\vec{a}\| + \vec{a}\cdot\vec{b}}{\|\vec{c}\|},$$

   the same thing.

10. In the triangle ABC, points A_1, B_1, and C_1 are chosen on the sides BC, AC, and AB respectively in such a way that AA_1, BB_1, and CC_1 are the three (angular) bisectors of the triangle ABC. Show that, if

$$\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1} = \vec{0}, \tag{1}$$

   then the triangle ABC must be equilateral.

   SOLUTION. Let's define the vectors $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$, $\overrightarrow{AB} = \vec{c}$, and let's denote their lengths by a, b, and c respectively. Then, we see immediately that

$$\vec{a} + \vec{b} + \vec{c} = \vec{0},$$

   and hence

$$\vec{c} = -\vec{a} - \vec{b}. \tag{2}$$

   Since the point C_1 is on the line segment AB, we must have

$$\overrightarrow{CC_1} = \lambda\overrightarrow{CA} + (1-\lambda)\overrightarrow{CB} = \lambda\vec{b} + (1-\lambda)(-\vec{a})$$

   for some real λ between 0 and 1, as shown in class. On the other hand, according to the result of the previous problem, $\overrightarrow{CC_1}$ must be proportional to

$$\|\overrightarrow{CB}\|\overrightarrow{CA} + \|\overrightarrow{CA}\|\overrightarrow{CB} = a\vec{b} + b(-\vec{a}).$$

   Combining these two observations, we conclude that

$$\overrightarrow{CC_1} = \frac{a}{a+b}\vec{b} - \frac{b}{a+b}\vec{a}.$$

   Similarly, we have

$$\overrightarrow{AA_1} = \frac{b}{b+c}\vec{c} - \frac{c}{b+c}\vec{b}$$

   and

$$\overrightarrow{CC_1} = \frac{c}{a+c}\vec{a} - \frac{a}{a+c}\vec{c}.$$

Adding these three equalities, and using (1), we get

$$\left(\frac{c}{a+c}-\frac{b}{a+b}\right)\vec{a}+\left(\frac{a}{a+b}-\frac{c}{b+c}\right)\vec{b}+\left(\frac{b}{b+c}-\frac{a}{a+c}\right)\vec{c}=\vec{0}.$$

Combining this with (2) we get, after simplification,

$$\left(\frac{b}{b+c}+\frac{b}{a+b}-1\right)\vec{a}=\left(\frac{a}{a+b}+\frac{a}{a+c}-1\right)\vec{b}.$$

Since the vectors $\vec{a}$ and $\vec{b}$ are not collinear, this is possible only if both coefficients are zeros, i.e.

$$\begin{aligned}\frac{b}{b+c}+\frac{b}{a+b} &= 1\\ \frac{a}{a+b}+\frac{a}{a+c} &= 1\end{aligned}$$

To solve this system, we get rid of the denominators, by multiplying the first equation by $(b+c)(a+b)$, and the second by $(a+b)(a+c)$. After simplification we get:

$$\begin{aligned}b^2 &= ac\\ a^2 &= bc\end{aligned}$$

or, equivalenly,

$$\begin{aligned}\frac{a}{b} &= \frac{b}{c}\\ \frac{a}{b} &= \frac{c}{a}.\end{aligned}$$

In other words, we have

$$\frac{a}{b}=\frac{b}{c}=\frac{c}{a},$$

which implies

$$a=b=c.$$