Dept. of Math. Sci., WPI
MA 1034 Analysis 4
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Homework Assignment 2

Solutions

1. Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be three vectors, such that $\vec{c} \neq \vec{0}$.

   (a) If $\vec{a} \cdot \vec{c} = \vec{b} \cdot \vec{c}$, does it follow that $\vec{a} = \vec{b}$? Explain.

   (b) If $\vec{a} \times \vec{c} = \vec{b} \times \vec{c}$, does it follow that $\vec{a} = \vec{b}$? Explain.

   (c) If $\vec{a} \cdot \vec{c} = \vec{b} \cdot \vec{c}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{c}$, does it follow that $\vec{a} = \vec{b}$? Explain.

   Solution. First of all, observe that the quality

   $$\vec{a} \cdot \vec{c} = \vec{b} \cdot \vec{c} \text{ is equivalent to } (\vec{a} - \vec{b}) \cdot \vec{c} = \vec{0},$$

   and the quality

   $$\vec{a} \times \vec{c} = \vec{b} \cdot \vec{c} \text{ is equivalent to } (\vec{a} - \vec{b}) \times \vec{c} = \vec{0}.$$

   (a) The answer is no. It simply means that $\vec{a} - \vec{b}$ is perpendicular to $\vec{c}$.

   (b) Again, the answer is no. It simply means that $\vec{a} - \vec{b}$ is parallel to $\vec{c}$.

   (c) Now the answer is yes. In this case, $\vec{a} - \vec{b}$ must be both perpendicular and parallel to $\vec{c}$. Since $\vec{c} \neq \vec{0}$, this is only possible if $\vec{a} - \vec{b} = \vec{0}$.

2. Let $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ be four vectors. Show that

   $$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{b} \cdot \vec{c} \\ \vec{a} \cdot \vec{d} & \vec{b} \cdot \vec{d} \end{vmatrix}. \qquad (1)$$

   Solution. This is a generalization of the equality

   $$\|\vec{a} \times \vec{b}\|^2 = \|\vec{a}\|^2 \|\vec{b}\|^2 - \|\vec{a} \cdot \vec{b}\|^2,$$

   or, equivalently,

   $$(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b}) = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{b} \cdot \vec{a} \\ \vec{a} \cdot \vec{a} & \vec{b} \cdot \vec{b} \end{vmatrix}, \qquad (2)$$

   which we considered in class.

   There are several ways to prove (1). A "brute-force" approach is to express all the four vectors in components, and verify the equality. This involves reducing an expression with 30 terms of the form $a_i b_j c_k d_l$.

   A more intelligent approach is the following. Look at both sides of (1). Each side is an expression of $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$. Each side is linear with respect to each of the four arguments. Thus, it suffices to prove (1) for the case when each of the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ is one of the coordinate vectors $\vec{i}$, $\vec{j}$, $\vec{k}$.

   Further, on both sides of (1), exchanging $\vec{a}$ and $\vec{b}$ changes the sign of the expression, and so does the exchanging of $\vec{c}$ and $\vec{d}$. On the other hand, exchanging the pair $\vec{a}$, $\vec{b}$ with the pair $\vec{c}$, $\vec{d}$, does not change either side of (1) at all.

Finally, any true equality involving only dot and cross products of $\vec{\imath}$, $\vec{\jmath}$, $\vec{k}$, will remain true if we replace $\vec{\imath}$, $\vec{\jmath}$, $\vec{k}$ cyclically by $\vec{\jmath}$, $\vec{k}$, $\vec{\imath}$, respectively.

The last two observations allow us to reduce the proof of (1) to only two cases:

If $\vec{a} = \vec{\imath}$, $\vec{b} = \vec{\jmath}$, $\vec{c} = \vec{\jmath}$, $\vec{d} = \vec{k}$, then both sides are equal to zero.

If $\vec{a} = \vec{c} = \vec{\imath}$, $\vec{b} = \vec{d} = \vec{\jmath}$, then both sides are equal to one.

All the other cases follow from these two, by using the above observations.

3. Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be three vectors.

   (a) Show that $\vec{a} \times (\vec{b} \times \vec{c})$ is a linear combination of $\vec{b}$ and $\vec{c}$. Use the result of the previous problem to find the coefficients of this linear combination.

   (b) Show that
$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0}$$

Solution.

(a) If $\vec{b}$ and $\vec{c}$ are collinear, then $\vec{a} \times (\vec{b} \times \vec{c})$ is zero, and the statement is obvious.

Let's look at the case when $\vec{b}$ and $\vec{c}$ are not collinear. By the geometric interpretation of cross product, the vector $\vec{a} \times (\vec{b} \times \vec{c})$ is perpendicular to $\vec{b} \times \vec{c}$. The latter vector, $\vec{b} \times \vec{c}$, is, in turn, perpendicular to both $\vec{b}$ and $\vec{c}$. Combining these two observations we see that $\vec{a} \times (\vec{b} \times \vec{c})$ must lie in the plane of $\vec{b}$ and $\vec{c}$, and hence be a linear combination of these two:
$$\vec{a} \times (\vec{b} \times \vec{c}) = \lambda \vec{b} + \mu \vec{c}.$$

To find λ and μ, we will dot-multyply the above equality by $\vec{b}$ and by $\vec{c}$:

$$\begin{aligned} \vec{b} \cdot (\vec{a} \times (\vec{b} \times \vec{c})) &= \lambda \vec{b} \cdot \vec{b} + \mu \vec{b} \cdot \vec{c} \\ \vec{c} \cdot (\vec{a} \times (\vec{b} \times \vec{c})) &= \lambda \vec{c} \cdot \vec{b} + \mu \vec{c} \cdot \vec{c}. \end{aligned}$$

Using the fact that, for any three vectors, $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$, we can rewrite the above pair of equations as

$$\begin{aligned} (\vec{b} \times \vec{a}) \cdot (\vec{b} \times \vec{c}) &= \lambda \vec{b} \cdot \vec{b} + \mu \vec{b} \cdot \vec{c} \\ (\vec{c} \times \vec{a}) \cdot (\vec{b} \times \vec{c}) &= \lambda \vec{c} \cdot \vec{b} + \mu \vec{c} \cdot \vec{c}. \end{aligned}$$

Next, we can appeal to the result of the previous problem, and recast the two equations into the form:

$$\begin{aligned} (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})(\vec{b} \cdot \vec{c}) &= \lambda \vec{b} \cdot \vec{b} + \mu \vec{b} \cdot \vec{c} \\ (\vec{a} \cdot \vec{c})(\vec{c} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})(\vec{c} \cdot \vec{c}) &= \lambda \vec{c} \cdot \vec{b} + \mu \vec{c} \cdot \vec{c}. \end{aligned}$$

This is a linear system of two equations for the two unknowns, λ and μ. One solution is readily visible: $\lambda = \vec{a} \cdot \vec{c}$, $\mu = -\vec{a} \cdot \vec{b}$. Since the vectors $\vec{b}$ and $\vec{c}$ are not collinear, the determinant of the system, $(\vec{b} \cdot \vec{b})(\vec{c} \cdot \vec{c}) - (\vec{b} \cdot \vec{c})(\vec{c} \cdot \vec{b})$ is non-zero; this means that there are no other solutions. Hence,

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}.$$

(b) Usung the result of the previous part, we can write

$$\begin{aligned}
\vec{a} \times (\vec{b} \times \vec{c}) &= (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \\
\vec{b} \times (\vec{c} \times \vec{a}) &= (\vec{b} \cdot \vec{a})\vec{c} - (\vec{b} \cdot \vec{c})\vec{a} \\
\vec{c} \times (\vec{a} \times \vec{b}) &= (\vec{c} \cdot \vec{b})\vec{a} - (\vec{c} \cdot \vec{a})\vec{b}
\end{aligned}$$

Adding these three equalities, we get the desired result,

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0}$$

This is called Jacobi's identity, and it shows that the vectors in $I\!R^3$, with the usual linear space operations, and with the cross product, form a *Lie algebra*.

4. Find the area of the triangle with vertices $P(1, -1, 2)$, $Q(3, 2, 3)$, and $R(0, 1, -1)$.

SOLUTION. Denote $\vec{a} = \overrightarrow{PQ} = \langle 2, 3, 1\rangle$ and $\vec{b} = \overrightarrow{PR} = \langle -1, 2, -3\rangle$. We have

$$\vec{a} \times \vec{b} = \langle -11, 5, 7\rangle,$$

and

$$\|\vec{a} \times \vec{b}\| = \sqrt{121 + 25 + 49} = \sqrt{195}$$

is the area of the parallelogram spanned by the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. The area of the triangle is half of the area of the parallelogram, i.e. $\sqrt{195}\,/2$.

5. Find an equation for the line which both passes through $P_0(-1, 2, 3)$ and

   (a) is parallel to the line defined by $l(t) = (1, 2t, -3 + t)$.

   (b) is perpendicular to the line defined by $l(t) = (1, 2t, -3 + t)$.

SOLUTION.

(a) Since the lines are apallel, they have the same direction vector, $\langle 0, 2, 1\rangle$. We are looking for a line passing through $(-1, 2, 3)$. Putting these two pieces of information together, we get the representation $(-1, 2 + 2t, 3 + t)$.

(b) Since the two lines are perpendicular, they must cross . More precisely, they cross at the intersection point of l and a plane perpendicular to l and passing through P. The normal vector of this plane is equal to the direction vetot of l, $\langle 0, 2, 1\rangle$. The equation of this plane is

$$2(y - 2) + (z - 3) = 0.$$

To find the intersection point Q, we substitute $x = -1$, $y = 2t$, $z = -3 + t$,

$$2(2t - 2) + (-3 + t - 3) = 0.$$

This gives $t = 2$, and Q has coordinates $(-1, 4, -1)$. The line we are looking for passes through the points P and Q. A direction vector is $\overrightarrow{PQ} = \langle 0, 2, -4\rangle$. The parametric equation is then $(-1 + 2t, 2 + 2t, 3 - 4t)$.

6. Find the distance from the point $P(1, -1, 2)$ to the line given by $l(t) = (1 + t, 2 - 2t, 3 + t)$.

SOLUTION. We start by choosing a point on the line l, say the point $M(1, 2, 3)$ corresponding to $t = 0$. Denote $\vec{u} = \overrightarrow{MP} = \langle 0, -3, -1\rangle$ and $\vec{v} = \langle 1, -2, 1\rangle$ (the direction vector of l). Let θ be the angle between $\vec{u}$ and $\vec{v}$. Then the distance from P to l is equal to $|MP|\sin\theta$. We have

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \frac{5}{\sqrt{10}\sqrt{6}},$$

and the distance is

$$|MP|\sin\theta = \|\vec{u}\|\sin\theta = \sqrt{10}\sqrt{1-\frac{25}{60}} = \sqrt{35/6}.$$

7. Consider the lines $l_1(t) = (3t+3, t+3, t)$ and $l_2(t) = (t-2, t, 2t-5)$.

   (a) Do these lines intersect?

   (b) If t represents time, and a particle travels on each line with its position determined by $l_1(t)$ and $l_2(t)$, will they ever collide?

   Solution.

   (a) The two lines intersect if they have a common point, i.e. if $l_1(t_1) = l_2(t_2)$ for some values of t_1 and t_2. In coordinates, this means

   $$\begin{aligned} 3t_1+3 &= t_2-2 \\ t_1+3 &= t_2 \\ t_1 &= 2t_2-5 \end{aligned}$$

   Solving this, we get $t_1 = -1$, $t_2 = 2$, and the intersection point is $l_1(-1) = l_2(2) = (0, 2, -1)$.

   (b) For the two particles to collide, they must be at the same place at the same time. This cannot happen, because, for example, the y-coordinates are always different (they differ by 3).

8. Find a point where the line given by $\left(t+1, 2t-1, \frac{t}{3}\right)$ intersects the plane $2x-y+3z=6$.

   Solution. The coordinates (x, y, z) of the intersection point must be both on the line and on the plane. This means

   $$\begin{aligned} x &= t+1 \\ y &= 2t-1 \\ z &= \frac{t}{3} \\ 2x-y+3z &= 6 \end{aligned}$$

   Solving this system, we get $t=3$, $x=4$, $y=5$, $z=1$. The intersection point is $(4,5,1)$.

9. Find the distance from the point $P(0,-1,2)$ to the plane $2x+3y-z=6$.

   Solution. The distance is given by the formula we derived in class:

   $$\frac{|(2)(0)+(3)(-1)-(2)-6|}{\sqrt{2^2+3^2+1^2}} = \frac{11}{\sqrt{14}}.$$

10. Find the (acute) angle beteween the planes $2x - y = 7$ and $-x + y - 3z = 5$.

Solution. The angle between the two planes is equal to the angle between their normal vectors, $\vec{u} = \langle 2, -1, 0 \rangle$, and $\vec{v} = \langle -1, 1, -3 \rangle$. In fact, there are two angles between the planes which complement each other to 2π. The cosines of these two angles have the same absolute value and opposite signs. The cosine corresponding to the acute angle θ is positive. We have

$$\cos\theta = \frac{|\vec{u} \cdot \vec{v}|}{\|\vec{u}\| \, \|\vec{v}\|} = \frac{3}{\sqrt{5}\sqrt{11}}.$$

The angle is $\arccos(3/\sqrt{55}) \approx 1.15434$, or, in degrees, approximately $66^o 8' 20''$.