1. Let $\vec{a}$, $\vec{b}$, and $\vec{c}$ be three vectors, such that $\vec{c} \neq \vec{0}$.

(a) If $\vec{a} \cdot \vec{c} = \vec{b} \cdot \vec{c}$, does it follow that $\vec{a} = \vec{b}$? Explain.

(b) If $\vec{a} \times \vec{c} = \vec{b} \times \vec{c}$, does it follow that $\vec{a} = \vec{b}$? Explain.

(c) If $\vec{a} \cdot \vec{c} = \vec{b} \cdot \vec{c}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{c}$, does it follow that $\vec{a} = \vec{b}$? Explain.

SOLUTION. First of all, observe that the quality $\vec{a} \cdot \vec{c} = \vec{b} \cdot \vec{c}$ is equivalent to $(\vec{a} - \vec{b}) \cdot \vec{c} = \vec{0}$, and the quality $\vec{a} \times \vec{c} = \vec{b} \times \vec{c}$ is equivalent to $(\vec{a} - \vec{b}) \times \vec{c} = \vec{0}$.

(a) The answer is no. It simply means that $\vec{a} - \vec{b}$ is perpendicular to $\vec{c}$.

(b) Again, the answer is no. It simply means that $\vec{a} - \vec{b}$ is parallel to $\vec{c}$.

(c) Now the answer is yes. In this case, $\vec{a} - \vec{b}$ must be both perpendicular and parallel to $\vec{c}$. Since $\vec{c} \neq \vec{0}$, this is only possible if $\vec{a} - \vec{b} = \vec{0}$.

2. Let $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ be four vectors. Show that

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \left| \begin{array}{ccc} \vec{a} \cdot \vec{c} & \vec{b} \cdot \vec{c} \\ \vec{a} \cdot \vec{d} & \vec{b} \cdot \vec{d} \end{array} \right|,$$  

(1)

SOLUTION. This is a generalization of the equality

$$||\vec{a} \times \vec{b}||^2 = ||\vec{a}||^2 ||\vec{b}||^2 - (\vec{a} \cdot \vec{b})^2,$$

or, equivalently,

$$(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b}) = \left| \begin{array}{ccc} \vec{a} \cdot \vec{a} & \vec{b} \cdot \vec{a} \\ \vec{a} \cdot \vec{b} & \vec{b} \cdot \vec{b} \end{array} \right|,$$  

(2)

which we considered in class.

There are several ways to prove (1). A “brute-force” approach is to express all the four vectors in components, and verify the equality. This involves reducing an expression with 30 terms of the form $a_i b_j c_k d_l$.

A more intelligent approach is the following. Look at both sides of (1). Each side is an expression of $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$. Each side is linear with respect to each of the four arguments. Thus, it suffices to prove (1) for the case when each of the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ is one of the coordinate vectors $\vec{i}$, $\vec{j}$, $\vec{k}$.

Further, on both sides of (1), exchanging $\vec{a}$ and $\vec{b}$ changes the sign of the expression, and so does the exchanging of $\vec{c}$ and $\vec{d}$. On the other hand, exchanging the pair $\vec{a}$, $\vec{b}$ with the pair $\vec{c}$, $\vec{d}$, does not change either side of (1) at all.
Finally, any true equality involving only dot and cross products of \( \vec{i}, \vec{j}, \vec{k} \), will remain true if we replace \( \vec{i}, \vec{j}, \vec{k} \) cyclically by \( \vec{j}, \vec{k}, \vec{i} \), respectively.

The last two observations allow us to reduce the proof of (1) to only two cases:

If \( \vec{a} = \vec{i}, \vec{b} = \vec{j}, \vec{c} = \vec{j}, \vec{d} = \vec{k} \), then both sides are equal to zero.

If \( \vec{a} = \vec{c} = \vec{i}, \vec{b} = \vec{d} = \vec{j} \), then both sides are equal to one.

All the other cases follow from these two, by using the above observations.

3. Let \( \vec{a}, \vec{b}, \) and \( \vec{c} \) be three vectors.

(a) Show that \( \vec{a} \times (\vec{b} \times \vec{c}) \) is a linear combination of \( \vec{b} \) and \( \vec{c} \). Use the result of the previous problem to find the coefficients of this linear combination.

(b) Show that

\[
\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0}
\]

Solution.

(a) If \( \vec{b} \) and \( \vec{c} \) are collinear, then \( \vec{a} \times (\vec{b} \times \vec{c}) \) is zero, and the statement is obvious.

Let’s look at the case when \( \vec{b} \) and \( \vec{c} \) are not collinear. By the geometric interpretation of cross product, the vector \( \vec{a} \times (\vec{b} \times \vec{c}) \) is perpendicular to \( \vec{b} \times \vec{c} \). The latter vector, \( \vec{b} \times \vec{c} \), is, in turn, perpendicular to both \( \vec{b} \) and \( \vec{c} \). Combining these two observations we see that \( \vec{a} \times (\vec{b} \times \vec{c}) \) must lie in the plane of \( \vec{b} \) and \( \vec{c} \), and hence be a linear combination of these two:

\[
\vec{a} \times (\vec{b} \times \vec{c}) = \lambda \vec{b} + \mu \vec{c}.
\]

To find \( \lambda \) and \( \mu \), we will dot-multyply the above equality by \( \vec{b} \) and by \( \vec{c} \):

\[
\begin{align*}
\vec{b} \cdot (\vec{a} \times (\vec{b} \times \vec{c})) &= \lambda \vec{b} \cdot \vec{b} + \mu \vec{b} \cdot \vec{c} \\
\vec{c} \cdot (\vec{a} \times (\vec{b} \times \vec{c})) &= \lambda \vec{c} \cdot \vec{b} + \mu \vec{c} \cdot \vec{c}.
\end{align*}
\]

Using the fact that, for any three vectors, \( \vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w} \), we can rewrite the above pair of equations as

\[
\begin{align*}
(\vec{b} \times \vec{a}) \cdot (\vec{b} \times \vec{c}) &= \lambda \vec{b} \cdot \vec{b} + \mu \vec{b} \cdot \vec{c} \\
(\vec{c} \times \vec{a}) \cdot (\vec{b} \times \vec{c}) &= \lambda \vec{c} \cdot \vec{b} + \mu \vec{c} \cdot \vec{c}.
\end{align*}
\]

Next, we can appeal to the result of the previous problem, and recast the two equations into the form:

\[
\begin{align*}
(\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})(\vec{b} \cdot \vec{c}) &= \lambda \vec{b} \cdot \vec{b} + \mu \vec{b} \cdot \vec{c} \\
(\vec{a} \cdot \vec{c})(\vec{c} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})(\vec{c} \cdot \vec{c}) &= \lambda \vec{c} \cdot \vec{b} + \mu \vec{c} \cdot \vec{c}.
\end{align*}
\]

This is a linear system of two equations for the two unknowns, \( \lambda \) and \( \mu \). One solution is readily visible: \( \lambda = \vec{a} \cdot \vec{c}, \mu = -\vec{a} \cdot \vec{b} \). Since the vectors \( \vec{b} \) and \( \vec{c} \) are not collinear, the determinant of the system, \( (\vec{b} \cdot \vec{b})(\vec{c} \cdot \vec{c}) - (\vec{a} \cdot \vec{b})(\vec{c} \cdot \vec{c}) \) is non-zero; this means that there are no other solutions. Hence,

\[
\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}.
\]
(b) Using the result of the previous part, we can write
\[ \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \]
\[ \vec{b} \times (\vec{c} \times \vec{a}) = (\vec{b} \cdot \vec{a})\vec{c} - (\vec{b} \cdot \vec{c})\vec{a} \]
\[ \vec{c} \times (\vec{a} \times \vec{b}) = (\vec{c} \cdot \vec{b})\vec{a} - (\vec{c} \cdot \vec{a})\vec{b} \]

Adding these three equalities, we get the desired result,
\[ \vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = \vec{0} \]

This is called Jacobi’s identity, and it shows that the vectors in \( \mathbb{R}^3 \), with the usual linear space operations, and with the cross product, form a Lie algebra.

4. Find the area of the triangle with vertices \( P(1, -1, 2) \), \( Q(3, 2, 3) \), and \( R(0, 1, -1) \).

**Solution.** Denote \( \vec{a} = \overrightarrow{PQ} = (2, 3, 1) \) and \( \vec{b} = \overrightarrow{PR} = (-1, 2, -3) \). We have
\[ \vec{a} \times \vec{b} = (-11, 5, 7), \]
and
\[ ||\vec{a} \times \vec{b}|| = \sqrt{121 + 25 + 49} = \sqrt{195} \]
is the area of the parallelogram spanned by the vectors \( \overrightarrow{PQ} \) and \( \overrightarrow{PR} \). The area of the triangle is half of the area of the parallelogram, i.e. \( \sqrt{195}/2 \).

5. Find an equation for the line which both passes through \( P_0(-1, 2, 3) \) and

(a) is parallel to the line defined by \( l(t) = (1 + t, 2t, -3 + t) \).

(b) is perpendicular to the line defined by \( l(t) = (1 + t, 2t, -3 + t) \).

**Solution.**

(a) Since the lines are parallel, they have the same direction vector, \( (0, 2, 1) \). We are looking for a line passing through \((-1, 2, 3)\). Putting these two pieces of information together, we get the representation \((-1, 2 + 2t, 3 + t)\).

(b) Since the two lines are perpendicular, they must cross. More precisely, they cross at the intersection point of \( l \) and a plane perpendicular to \( l \) and passing through \( P \). The normal vector of this plane is equal to the direction vector of \( l \), \( (0, 2, 1) \). The equation of this plane is
\[ 2(y - 2) + (z - 3) = 0. \]

To find the intersection point \( Q \), we substitute \( x = -1, y = 2t, z = -3 + t \),
\[ 2(2t - 2) + (-3 + t - 3) = 0. \]

This gives \( t = 2 \), and \( Q \) has coordinates \((-1, 4, -1)\). The line we are looking for passes through the points \( P \) and \( Q \). A direction vector is \( \overrightarrow{PQ} = (0, -2, -4) \). The parametric equation is then \((-1 + 2t, 2 + 2t, 3 - 4t)\).

6. Find the distance from the point \( P(1, -1, 2) \) to the line given by \( l(t) = (1 + t, 2 - 2t, 3 + t) \).

**Solution.** We start by choosing a point on the line \( l \), say the point \( M(1, 2, 3) \) corresponding to \( t = 0 \). Denote \( \vec{u} = \overrightarrow{MP} = (0, -3, -1) \) and \( \vec{v} = (1, -2, 1) \) (the direction vector
Let $\theta$ be the angle between $\vec{u}$ and $\vec{v}$. Then the distance from $P$ to $l$ is equal to $|MP|\sin \theta$. We have
\[
\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \frac{5}{\sqrt{10\sqrt{10}}}.
\]
and the distance is
\[
|MP|\sin \theta = \|\vec{u}\| \sin \theta = \sqrt{10\sqrt{10}} \sqrt{1 - \frac{25}{60}} = \sqrt{35/6}.
\]

7. Consider the lines $l_1(t) = (3t + 3, t + 3, t)$ and $l_2(t) = (t - 2, t, 2t - 5)$.

(a) Do these lines intersect?
(b) If $t$ represents time, and a particle travels on each line with its position determined by $l_1(t)$ and $l_2(t)$, will they ever collide?

**Solution.**

(a) The two lines intersect if they have a common point, i.e. if $l_1(t_1) = l_2(t_2)$ for some values of $t_1$ and $t_2$. In coordinates, this means
\[
\begin{align*}
3t_1 + 3 &= t_2 - 2 \\
t_1 + 3 &= t_2 \\
t_1 &= 2t_2 - 5
\end{align*}
\]
Solving this, we get $t_1 = -1$, $t_2 = 2$, and the intersection point is $l_1(-1) = l_2(2) = (0, 2, -1)$.

(b) For the two particles to collide, they must be at the same place at the same time. This cannot happen, because, for example, the $y$-coordinates are always different (they differ by 3).

8. Find a point where the line given by $(t + 1, 2t - 1, \frac{t}{3})$ intersects the plane $2x - y + 3z = 6$.

**Solution.** The coordinates $(x, y, z)$ of the intersection point must be both on the line and on the plane. This means
\[
\begin{align*}
x &= t + 1 \\
y &= 2t - 1 \\
z &= \frac{t}{3} \\
2x - y + 3z &= 6
\end{align*}
\]
Solving this system, we get $t = 3$, $x = 4$, $y = 5$, $z = 1$. The intersection point is $(4, 5, 1)$.

9. Find the distance from the point $P(0, -1, 2)$ to the plane $2x + 3y - z = 6$.

**Solution.** The distance is given by the formula we derived in class:
\[
\frac{|(2)(0) + (3)(-1) - (2) - 6|}{\sqrt{2^2 + 3^2 + 1^2}} = \frac{11}{\sqrt{14}}.
\]
10. Find the (acute) angle between the planes $2x - y = 7$ and $-x + y - 3z = 5$.

**Solution.** The angle between the two planes is equal to the angle between their normal vectors, $\vec{u} = \langle 2, -1, 0 \rangle$, and $\vec{v} = \langle -1, 1, -3 \rangle$. In fact, there are two angles between the planes which complement each other to $2\pi$. The cosines of these two angles have the same absolute value and opposite signs. The cosine corresponding to the acute angle $\theta$ is positive. We have

$$\cos \theta = \frac{|\vec{u} \cdot \vec{v}|}{\|\vec{u}\| \|\vec{v}\|} = \frac{3}{\sqrt{55}}.$$

The angle is $\arccos(3/\sqrt{55}) \approx 1.15434$, or, in degrees, approximately $66^\circ 8' 20''$. 