Dept. of Math. Sci., WPI
MA 1034 Analysis 4
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Homework Assignment 3

Solutions

1. Let $A, B \subseteq I\!R^n$. Prove that $(\text{int}A) \cap (\text{int}B) = \text{int}(A \cap B)$. Is the statement true if "intersection" is replaced by "union"? Explain.

   Solution. To show that two sets are equal means to show that they have the same elements.

   Let $x \in (\text{int}A) \cap (\text{int}B)$. This means $x \in \text{int}A$ and $x \in \text{int}B$. Therevore, there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ so that $B_{\varepsilon_1}(x) \subseteq A$ and $B_{\varepsilon_2}(x) \subseteq B$. Choose $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Then $B_\varepsilon(x) \subseteq A$ and $B_\varepsilon(x) \subseteq B$, hence $B_\varepsilon(x) \subseteq A \cap B$. This, in turn, means $x \in \text{int}(A \cap B)$.

   Conversely, let now $x \in \text{int}(A \cap B)$. This means that, for some $\varepsilon > 0$, we have $B_\varepsilon(x) \subseteq A \cap B$. This implies $B_\varepsilon(x) \subseteq A$ and $B_\varepsilon(x) \subseteq B$. This, in turn, means $x \in \text{int}A$ and $x \in \text{int}B$, i.e. $x \in (\text{int}A) \cap (\text{int}B)$.

   If "intersection" is replaced by "union", we can show easily that

   $$(\text{int}A) \cup (\text{int}B) \subseteq \text{int}(A \cup B).$$

   However, the converse inclusion is not true in general, as the following example shows. Let $A = \{(x, y) : x \leq 0\}$ and $B = \{(x, y) : x \geq 0\}$. Then the point $(0, 0)$ is in $\text{int}(A \cup B)$, but not in $(\text{int}A) \cup (\text{int}B)$.

2. Let $A, B \subseteq I\!R^n$. Prove that $(A' \cup B') = (A \cup B)'$. Is the statement true if "union" is replaced by "intersection"? Explain.

   Solution. Again, to show that two sets are equal means to show that they have the same elements.

   Let $x \in (A' \cup B')$. This means $x \in A'$ or $x \in B'$. By symmetry, it is enough to consider the case $x \in A'$. There exists a sequence $x_1, x_2, \cdots$ so that, for each n, we have $x_n \neq x$, $x_n \in A$, and $x_n \to x$ as $n \to \infty$. Since $x_n \in A \subseteq A \cup B$, we see that $x \in (A \cup B)'$.

   Now, conversely, let $x \in (A \cup B)'$. This means that there exists a sequence $x_1, x_2, \cdots$ so that, for each n, we have $x_n \neq x$, $x_n \in (A \cup B)$, and $x_n \to x$ as $n \to \infty$. Observe that $x_n \in (A \cup B)$ means $x_n \in A$ or $x_n \in B$. At least one of the sets A and B must contain infinitely many terms of the sequence. Then x is an accumulation point for that set, and therefore $x \in (A' \cup B')$.

   Again, if "union" is replaced by "intersection", we can easily show that

   $$(A' \cap B') \supseteq (A \cap B)',$$

   but the converse inclusion is not true in general, as the following example shows. Let $A = \{(x, y) : x \leq 0, |y| \leq |x|\}$ and $B = \{(x, y) : x \geq 0, |y| \leq x\}$. Then the point $(0, 0)$ is in $A' \cap B'$, but not in $(A \cap B)'$.

3. Show that the function $f(x, y) = \frac{1}{x+y}$ is continuous but not uniformly continuous on the open square $D = (0, 1) \times (0, 1)$.

SOLUTION. Consider the two sequences $(x_n, y_n) = (1/n, 1/n)$ and $(s_n, t_n) = (1/(n+1), 1/(n+1))$. We have $\sqrt{(s_n - x_n)^2 + (t_n - y_n)^2} \to 0$ as $n \to \infty$, but

$$|f(s_n, t_n) - f(x_n, y_n)| = \frac{1}{2} \not\to 0$$

as $n \to \infty$.

In problems 4-7, find the limit, if it exists, or show that the limit does not exist.

4.

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2}.$$

SOLUTION. The limit does not exist. Indeed, if we take $(x_n, y_n) = (1/n, 1/n)$, we get a limit $1/2$, but if we take $(x_n, y_n) = (1/n, 0)$, we get a limit 1.

5.

$$\lim_{(x,y)\to(0,0)} \frac{xy \cos y}{3x^2 + y^2}.$$

SOLUTION. The limit does not exist. Indeed, if we take $(x_n, y_n) = (1/n, 1/n)$, we get a limit $1/4$, but if we take $(x_n, y_n) = (1/n, 0)$, we get a limit 0.

6.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2 + 2y^2}.$$

SOLUTION. The limit is zero. Indeed, we have

$$\frac{x^2 \sin^2 y}{x^2 + 2y^2} = \frac{x^2}{x^2 + 2y^2} \sin^2 y.$$

The first term of this product is bounded, while the second one, $\sin^2 y$ goes to zero as $(x, y) \to (0, 0)$.

7.

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz + xz}{x^2 + y^2 + z^2}.$$

SOLUTION. The limit does not exist. Indeed, if we take $(x_n, y_n, z_n) = (1/n, 1/n, 1/n)$, we get a limit 1, but if we take $(x_n, y_n, z_n) = (1/n, 1/n, 0)$, we get a limit $1/2$.

In problems 8-10, determine the set of points where the function is continuous.

8.

$$F(x, y) = \frac{\sin(xy)}{e^x - y^2}$$

SOLUTION. The given function is the ratio of two continuous functions and is continuous wherever it is defined, i.e. on the set

$$\{(x, y) : e^x \neq y^2\}.$$

9.

$$f(x,y,z) = \frac{\sqrt{y}}{x^2 - y^2 + z^2}$$

Solution. The given function is the ratio of a squre root and a polynomial. It will be continuous whrever the square root is defined and the polynomial is not zero. In other words, f will be continuous throughout its domain, i.e., the set

$$\{(x,y,z) : y > 0, y^2 \neq x^2 + z^2\}.$$

10.

$$f(x,y,z) = \arcsin\sqrt{x^2 + y^2 + z^2}$$

Solution. The given function is a composition of the arcsine, a square root, and a polynomial. Each of these functions is continuous wherever it is defined. Their composition will be continuous throughout the domain of the function, i.e., the set

$$\{(x,y,z) : x^2 + y^2 + z^2 \leq 1\}.$$