Dept. of Math. Sci., WPI
MA 1034 Analysis 4
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F I N A L   E X A M

Due Thursday, May 1, 2003, by 5:00pm

INSTRUCTIONS

- This is a take-home exam. You must work on your own.
- Each problem has equal weight (20 points).
- You must turn in the solutions to 5 (five) of the following problems. If you turn in more, only your first five will be graded.
- You can turn for questions and clarifications to me, but you cannot discuss the exam with anyone else.
- You can freely use the textbook and your classnotes, as well as the posted solutions to the homework assignments and the midsemester exam.
- If you use any other text, you must indicate this in your solution, and give a clear reference to the text you have used, and to what extent.

GOOD LUCK!

1. Find the acute angle between two diagonals of a cube.

2. In the triangle ABC, $|AB| = |BC|$. A point D is chosen on the side BC so that $|BD| : |DC| = 1 : 4$ and a point E is chosen on the side AC so that BE is perpendicular to AC. In what ratio does the line AD divide the segment BE?

3. Let $A_1A_2 \ldots A_n$ be a regular n-gon, inscribed in a circle with center O and radius r. Let X be an arbitrary point on the plane and denote $d = |OX|$. Show that

$$\sum_{i=1}^{n} |A_iX|^2 = n(r^2 + d^2).$$

4. Let P be a point not on the plane that passes through the points Q, R, and S. Show that the distance d from P to the plane is

$$d = \frac{|(\vec{a} \times \vec{b}) \cdot \vec{c}|}{|\vec{a} \times \vec{b}|},$$

where $\vec{a} = \overrightarrow{QR}$, $\vec{b} = \overrightarrow{QS}$, and $\vec{c} = \overrightarrow{QP}$.

5. Show that the curve of intersection of the surfaces $x^2+2y^2-z^2+3x = 1$ and $2x^2 + 4y^2 - 2z^2 - 5y = 0$ lies in a plane, and find an equation of this plane.

6. A vector-valued function $\vec{r}(t)$, defined for $t \in [a, b]$ is called Lipschitz-continuous if there exists a constant L such that, for any choice of $t_1, t_2 \in [a, b]$, we have

$$|\vec{r}(t_2) - \vec{r}(t_1)| \leq L|t_2 - t_1|.$$

If $\vec{r}(t)$ is Lipschitz-continuous, show that the curve parametrized by it is rectifiable.

7. Find the lengths of the curves described by:

   (a) $\vec{r}(t) = \langle 4\cos t, 4\sin t, 3t\rangle$, $0 \leq t \leq \pi$,

   (b) $\vec{r}(t) = \langle \sin(3t), \cos(3t), 2t^{3/2}\rangle$,$0 \leq t \leq 3$,

   (c) $\vec{r}(t) = \langle 3\sin t, 5\cos t, 4\sin t\rangle$, $0 \leq t \leq 1$.

8. Prove or disprove: if $\vec{u}$ and $\vec{v}$ are Riemann-integrable vector-valued functions on $[a, b]$, then

$$\int_a^b [\vec{u}(t) \cdot \vec{v}(t)]\, dt = \int_a^b \vec{u}(t)\, dt \cdot \int_a^b \vec{v}(t)\, dt.$$

9. Suppose that $w = x^3y + y^2 - 4z^3$, $x = rs$, $y = (r^2 - s^2)^3$, and $z = 2r + s$. Use the chain rule to compute $\frac{\partial w}{\partial s}$. Express your answer in terms of r and s alone. Algebraic simplification is not necessary.

10. For

$$f(x, y, z) = \frac{x}{x^2 + y^2} e^{\sin(x^2 y)} + \sin(z - 2) \ln(x + \sqrt{1 + x^2}),$$

compute $f_x(1, 0, 2)$.
Hint: There is an easy way to do this. Recall the definition.

11. Let f be a continuously differentiable function of one variable and define

$$w(x, y) := f\left(\frac{y}{y - x}\right).$$

Show that, for $x \neq y$:

$$x\, \frac{\partial w}{\partial x} + y\, \frac{\partial w}{\partial y} = 0.$$

12. Assume that u and v have continuous second partial derivatives and that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{and} \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

13. Let f be a smooth function with the following property: At the point (1,2,3), the function f decreases most rapidly in the direction given by the vector $\langle 5, -3, 0 \rangle$ and the rate of change of f in this direction is 17. Find $\nabla f(1, 2, 3)$.

14. Let R be the region described by

$$0 \le y \le \sqrt{16 - x^2},\ 0 \le x \le 4.$$

Find the absolute maximum and absolute minimum values of

$$f(x,y) = x^2 + y^2 - 4x - 2y + 1$$

on R.

15. Let $f(x,y)$ be defined and continuous for all $(x,y) \in \mathbb{R}^2$. For every $t > 0$ consider the region

$$D_t := \{(x,y) : x^2 + y^2 \le t^2\}$$

and define

$$F(t) := \iint_{D_t} f(x,y)\,dA.$$

Show that, for $t > 0$, the derivative of F is given by

$$F'(t) = \int_0^{2\pi} t\,f(t\cos\theta, t\sin\theta)\,d\theta.$$

16. Let f be continuous on $[0,1]$ and let R be the triangular region with vertices $(0,0), (1,0)$, and $(0,1)$. Show that

$$\iint_R f(x+y)\,dA = \int_0^1 u f(u)\,du.$$

17. Prove that

$$\int_0^1 \frac{x-1}{\ln x}\,dx = \ln 2.$$

Hint: Consider the function $f(x,y) = x^{y-1}$.

18. Prove that

$$\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx = \frac{\pi}{8}\ln 2.$$