Department of Mathematical Sciences
WPI
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Term D03 S O L U T I O N S

MA1034 MIDTERM EXAM
50 minutes

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	

- This is a 50-minute exam. It has five (5) problems. It is your responsibity to make sure you have all the pages.
- Write your solutions in the space provided below each problem. If you need more space, use the back of the sheet (please indicate clearly if you do so).
- No books or notes are to be consulted.
- Calculators are NOT allowed.
- The solutions you submit must be your own work. You may not look at or copy the work of others.
- Show all your work. No credit will be given for unsupported answers.

GOOD LUCK

1. (20 pts.) Let $\vec{a}$ and $\vec{b}$ be two vectors such that $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular. Show that $\|\vec{a}\| = \|\vec{b}\|$.

   SOLUTION.

$$\begin{aligned}
(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) &= 0\\
\vec{a}\cdot\vec{a}-\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}-\vec{b}\cdot\vec{b} &= 0\\
\vec{a}\cdot\vec{a}-\vec{b}\cdot\vec{b} &= 0\\
\vec{a}\cdot\vec{a} &= \vec{b}\cdot\vec{b}\\
\|\vec{a}\|^2 &= \|\vec{b}\|^2\\
\|\vec{a}\| &= \|\vec{b}\|
\end{aligned}$$

2. (20 pts.) Show that if $\vec{u}$, $\vec{v}$, and $\vec{w}$ are three-dimensional vectors such that

$$\vec{u} + \vec{v} + \vec{w} = 0,$$

then

$$\vec{u} \times \vec{v} = \vec{v} \times \vec{w} = \vec{w} \times \vec{u}.$$

SOLUTION. From the equality

$$\vec{u} + \vec{v} + \vec{w} = 0,$$

we get

$$\vec{u} = -\vec{v} - \vec{w} \qquad , \vec{v} = -\vec{w} - \vec{u} \qquad , \vec{w} = -\vec{u} - \vec{v}.$$

Using the properties of the cross-product we have:

$$\vec{u} \times \vec{v} = (-\vec{v} - \vec{w}) \times \vec{v} = -\vec{v} \times \vec{v} - \vec{w} \times \vec{v} = \vec{v} \times \vec{w},$$

and

$$\vec{u} \times \vec{v} = \vec{u} \times (-\vec{w} - \vec{u}) = -\vec{u} \times \vec{w} - \vec{u} \times \vec{u} = \vec{w} \times \vec{u}.$$

3. (20 pts.) Find the area of the triangle with vertices $P(2,-3,7)$, $Q(4,-1,8)$, and $R(3,-4,9)$.

SOLUTION. This is similar to problem 4 of homework 2:

$$\begin{aligned}\overrightarrow{PQ} &= \langle 2,2,1\rangle \\ \overrightarrow{PR} &= \langle 1,-1,2\rangle \\ \overrightarrow{PQ}\times\overrightarrow{PR} &= \langle 5,-3,4\rangle \\ \|\overrightarrow{PQ}\times\overrightarrow{PR}\| &= \sqrt{50}\end{aligned}$$

and the area is

$$\frac{1}{2}\|\overrightarrow{PQ}\times\overrightarrow{PR}\| = \frac{5}{\sqrt{2}}.$$

4. (20 pts.) A plane passes through the points $A(1,4,1)$, $B(4,7,-5)$, and $C(-1,5,8)$. A line passes through the points $P(3,-2,7)$ and $Q(6,-3,6)$. Find the intersection point of the line and the plane.

SOLUTION. First, we find the equation of the plane:

$$\begin{aligned} \overrightarrow{AB} &= \langle 3,3,-6\rangle \\ \overrightarrow{AC} &= \langle -2,1,7\rangle \\ \overrightarrow{AB}\times\overrightarrow{AC} &= \langle 27,-9,9\rangle, \end{aligned}$$

which is a normal vector. The equation of the plane is then

$$27(x-1)-9(y-4)+9(z-1)=0,$$

or, equivalently, after simplification,

$$3x-y+z=0.$$

Second, we find the parametric equation of the line. The direction vector is

$$\overrightarrow{PQ}=\langle 3,-1,-1\rangle,$$

and the parametric equation is $l(t)=(3+3t,-2-t,7-t)$.

Plugging these back into the equation of the plane, we get

$$3(3+3t)-(-2-t)+(7-t)=0,$$

which gives $t=-2$, $x=-3$, $y=0$, $z=9$. So, the intersection point has coordinates $(-3,0,9)$

5. (20 pts.) Find the distance between the plane

$$3x - 4z = -9$$

and the sphere

$$x^2 + y^2 + z^2 - 2x + 4y + 4z = 0.$$

Hint: A sphere is just a bloated point.

SOLUTION. Rewriting the equation of the sphere as

$$(x-1)^2 + (y+2)^2 + (z+2)^2 = 9,$$

we see that the sphere has a center $C(1, -2, -2)$, and radius $r = 3$. The distance between the plane and the the center C is

$$\frac{|(3)(1) - (4)(-2) + 9|}{\sqrt{3^2 + 4^2}} = \frac{20}{5} = 4.$$

Diminishing this by the radius $r = 3$, we obtain the distance between the plane and the sphere, $4 - 3 = 1$.