Dept. of Math. Sci., WPI
MA 3831 Advanced Calculus - I
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Homework Assignment 1

Solutions

Problem 1. Use induction to prove that, for every positive integer n, the number $7^n - 4^n$ is divisible by 3.

SOLUTION.

base: For $n = 1$, we have

$$7^1 - 4^1 = 3.$$

step: Assume that for some n,

$$7^n - 4^n = 3k$$

for some $k \in I\!N$. Then, for $(n + 1)$, we have

$$\begin{aligned} 7^{n+1} - 4^{n+1} &= (7) \cdot 7^n - (4) \cdot 4^n \\ &= (3) \cdot 7^n + 4 \cdot (7^n - 4^n) \\ &= 3(7^n + 4k) = 3m \end{aligned}$$

where $m = (7^n + 4k) \in I\!N$.

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Problem 2. Let $P(n)$ denote the statement:

$$1 + 2 + 3 + \cdots + n = \frac{1}{8}(2n + 1)^2.$$

(a) Prove that, if $P(n)$ is true for an integer n, then $P(n + 1)$ is also true.

(b) Criticize the statement: "By induction, $P(n)$ is true for all n."

(c) Amend $P(n)$ by changing the equality into an inequality that is true for all positive integers n.

SOLUTION. Observe that

$$\begin{aligned}
&\frac{1}{8}(2n+1)^2 + (n+1) \\
&= \frac{(2n+1)^2 + 8n + 8}{8} \\
&= \frac{(2n+1)^2 + 2(2n+1)\cdot 2 + 2^2}{8} \\
&= \frac{(2n+3)^2}{8} = \frac{(2(n+1)+1)^2}{8}
\end{aligned}$$

(a) Assuming that $P(n)$ is true for some n, we have for $(n+1)$:

$$1 + 2 + 3 + \cdots + n + (n+1) = \frac{1}{8}(2n+1)^2 + (n+1) = \frac{2(n+1)+1}{8},$$

i.e., $P(n+1)$ follows from $P(n)$.

(b) For a proof by induction, we have to chek the base of induction, $P(1)$, and this we did not do. Furthermore, it is easy to see that

$$1 \neq \frac{9}{8} = \frac{1}{8}(2\cdot(1)+1)^2,$$

i.e., $P(1)$ fails.

Thus, not only is the "proof" flawed, but the statement is also wrong.

(c) The right statement is:

$$1 + 2 + \cdots + n < \frac{1}{8}(2n+1)^2.$$

For $n = 1$, $1 < \frac{9}{8}$. If true for n, we have for $n+1$:

$$1 + 2 + 3 + \cdots + n + (n+1) < \frac{1}{8}(2n+1)^2 + (n+1) = \frac{2(n+1)+1}{8}.$$

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Problem 3. For real numbers x, we defined in class $[\![x]\!]$ as the unique integer such that

$$[\![x]\!] \leq x < [\![x]\!] + 1.$$

Prove the following properties:

(a) $[\![x+n]\!] = [\![x]\!] + n$ for every integer n.

(b) $[\![-x]\!] = \begin{cases} -[\![x]\!], & \text{if } x \text{ is an integer} \\ -[\![x]\!] - 1, & \text{if } x \text{ is not an integer} \end{cases}$

(c) $[\![x+y]\!]$ is equal to $[\![x]\!] + [\![y]\!]$ or $[\![x]\!] + [\![y]\!] + 1$.

(d) $[\![2x]\!] = [\![x]\!] + [\![x + \frac{1}{2}]\!]$

(e) $[\![3x]\!] = [\![x]\!] + [\![x + \frac{1}{3}]\!] + [\![x + \frac{2}{3}]\!]$

SOLUTION

(a) Let an arbitrary integer n be given. By definition,

$$[\![x]\!] \leq x < [\![x]\!] + 1.$$

Adding n:

$$[\![x]\!] + n \leq x + n < [\![x]\!] + n + 1.$$

Since $[\![x]\!] + n$ is an integer, the above inequalities show that

$$[\![x+n]\!] = [\![x]\!] + n$$

(b) If x is integer, then so is $-x$. Then

$$[\![x]\!] = x, [\![-x]\!] = -x.$$

Combining these two, we see that

$$[\![-x]\!] = -x = -[\![-x]\!].$$

Now, if x is not an integer, then

$$[\![x]\!] < x < [\![x]\!] + 1.$$

Multiply both sides by -1, reversing the inequality:

$$-[\![x]\!] > -x > -[\![x]\!] - 1.$$

This can be rewritten as

$$-[\![x]\!] - 1 < x < (-[\![x]\!] - 1) + 1,$$

which exactly means that

$$[\![-x]\!] = -[\![x]\!] - 1.$$

(c) Again, by definition,

$$[\![x]\!] \leq x < [\![x]\!] + 1,$$
$$[\![y]\!] \leq y < [\![y]\!] + 1.$$

Adding these two, we get:

$$[\![x]\!] + [\![y]\!] \leq x + y < [\![x]\!] + [\![y]\!] + 2.$$

Two cases are possible:

case $x + y < [\![x]\!] + [\![y]\!] + 1$. Then

$$[\![x]\!] + [\![y]\!] \leq x + y < [\![x]\!] + [\![y]\!] + 1$$

and

$$[\![x]\!] + [\![y]\!] = [\![x + y]\!].$$

case $x + y \geq [\![x]\!] + [\![y]\!] + 1$. Then

$$[\![x]\!] + [\![y]\!] + 1 \leq x + y < [\![x]\!] + [\![y]\!] + 2$$

and hence

$$[\![x]\!] + [\![y]\!] = [\![x + y]\!] + 1.$$

(d) By definition,

$$[\![x]\!] \leq x < [\![x]\!] + 1.$$

Again, two cases are possible.

case $[\![x]\!] \leq x < [\![x]\!] + \frac{1}{2}$. Then

$$2[\![x]\!] \leq 2x < 2[\![x]\!] + 1 \text{ and } [\![x]\!] + \frac{1}{2} \leq x + \frac{1}{2} < [\![x+]\!] + 1,$$

hence

$$[\![2x]\!] = 2[\![x]\!] \text{ and } [\![x + \frac{1}{2}]\!] = [\![x]\!].$$

Thus,

$$[\![2x]\!] = [\![x]\!] + [\![x + \frac{1}{2}]\!].$$

case $[\![x]\!] + \frac{1}{2} \leq x < [\![x]\!] + 1$. Then

$$2[\![x]\!] + 1 \leq 2x < 2[\![x]\!] + 2 \text{ and } [\![x]\!] + 1 \leq x + \frac{1}{2} < [\![x+]\!] + \frac{3}{2},$$

hence

$$[\![2x]\!] = 2[\![x]\!] + 1 \text{ and } [\![x + \frac{1}{2}]\!] = [\![x]\!] + 1.$$

Thus, again

$$[\![2x]\!] = [\![x]\!] + [\![x + \frac{1}{2}]\!].$$

(e) This is similar to (d), but we have 3 cases to cosider:

$$[\![x]\!] + \leq x < [\![x]\!] + \frac{1}{3}, [\![x]\!] + \frac{1}{3} \leq x < [\![x]\!] + \frac{2}{3}, [\![x]\!] + \frac{2}{3} \leq x < [\![x]\!] + 1.$$

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Problem 4. Let $S \subset \mathbb{R}$, $T \subset \mathbb{R}$ be non-empty and bounded above. Prove or disprove:

(a) $\sup(S \cup T) = \max\{\sup S, \sup T\}$

(b) $\sup(S \cap T) = \min\{\sup S, \sup T\}$

SOLUTION

(a) The equality is true. Let $b := \max\{\sup S, \sup T\}$. Since every element of $S \cup T$ is in S or T, and b is the greater of the numbers $\sup S$ and $\sup T$, b is an upper bound of $S \cup T$.

Next, we must show that it is the *least* upper bound. Let $a < b$ be given. Since b is one of the numbers $\sup S$ and $\sup T$, the number a is strictly less than one of the numbers $\sup S$ and $\sup T$. In other words, we can find an element in S or T that is greater than a. This means that a is not an upper bound of $S \cup T$.

(b) This one is wrong, as can be seen from the following example. Let $S = \{1, 2, 3\}, T = \{0, 2, 4\}$. Then

$$\sup(S \cap T) = 2 \neq 3 = \min\{3, 4\} = \min\{\sup S, \sup T\}.$$

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Problem 5. Let $x_1, x_2, \ldots, x_n, \ldots$ be a list of *positive* reals. Prove that if the set

$$S = \left\{ z : z = \sum_{k=1}^{n} x_k \text{ for some } n \in \mathbb{N} \right\}$$

is bounded above then there is exactly one number L with the following property:

For each $h > 0$, there are at most finitely many $z \in S$ **not** satisfying the inequality

$$L - h \leq z \leq L.$$

Solution

Uniqueness. Let L' be two real numbers with the property that, for every $h > 0$,

$$L - h \leq z \leq L, \text{ and } L' - h \leq z \leq L'$$

is true for all $z \in S$ except finitely many z's.

We will show that $L' = L$ by excluding the other two possibilities. Assume, for purposes of contoversy, that $L' > L$. Choose $h := (L' - L)/2 > 0$. Then, by the properties of L', there can be at most finitely many $z \in S$ for which $z < L' - h$, and, since $L < L' - h$, at most finitely many $z \in S$ for which $z \leq L$. This contradicts the properties of L. The case $L > L'$ is symmetric.

Existence. S is certainly nonempty (e.g., $x_1 \in S$). Since S is bounded above, there exists $\sup S$, denote

$$L := \sup S.$$

Since L is an upper bound, then

$$z \leq L \text{ for all } z \in S.$$

Now let $h > 0$ be given. Since L is the *least* upper bnound of S, $L - h$ is not an upper bound. This means there exists some $z_0 \in S$, such that $L - h < z_0$.

On the other hand, $z_0 \in S$ means that

$$z_0 = \sum_{k=1}^{n_0}$$

for some $n_0 \in I\!N$. Since all x_k are positive, then, for all $n \geq n_0$,

$$\sum_{k=1}^{n} \geq \sum_{k=1}^{n_0} = z_0 > L - h.$$

In other words, if some $z \in S$ violates

$$L - h < x \leq L,$$

then z must have the form

$$z = \sum_{k=1}^{n}$$

with $n < n_0$. Thus, there are at most finitely many such z.

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