Dept. of Math. Sci., WPI
MA 3831 Advanced Calculus - I
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Homework Assignment 2

Solutions

Problem 1. For this problem, assume without proof that there exists an irrational number $c \in (0,1)$. Using this fact and the properties of rational and irrational numbers that we proved in class:

(a) Show that, for any two rational numbers r_1, r_2 with $r_1 < r_2$, there exists an irrational number z such that $r_1 < z < r_2$.

(b) Show that, for any two real numbers x, y with $x < y$, there exists an irrational number z such that $x < z < y$.

SOLUTION.

(a) Take $z = r_1 + (r_2 - r_1)c$.

(b) Find a rational number $r_1 \in (x, y)$. Find a rational number $r_2 \in (r_1, y)$. Use part (a) to find an irrational $z \in (r_1, r_2) \subseteq (x, y)$.

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Problem 2. Show that $||x| - |y|| \leq |x - y|$, assuming the triangle inequality.

SOLUTION. According to the triangle inequality, for any two real numbers a and b,

$$|a + b| \leq |a| + |b|.$$

Taking $a = x - y$, and $b = y$, we get

$$|x| \leq |x - y| + |y|,$$

and taking $a = y - x$, and $b = x$, we get

$$|y| \leq |y - x| + |x|.$$

Rearranging these two inequalities, we can write them as:

$$\begin{aligned} |x| - |y| &\leq |x - y| \\ |y| - |x| &\leq |x - y|. \end{aligned}$$

Since $||x| - |y||$ is one of the numbers $|x| - |y|$ and $|y| - |x|$, we get the desired inequality.

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Problem 3. Under what condition is it true that

$$|x - y| + |y - z| = |x - z|?$$

SOLUTION. Let's denote for a moment $a := x - y$, $b := y - z$. Then the equalty in question becomes $|a| + |b| = |a + b|$. This is obviously true if at least one of the numbers a and b is 0. In what other cases will it be true? There are four possibilities regarding the signs of a and b that we must consider.

case $a > 0, b > 0$. Then $a + b > 0$ and

$$|a| + |b| = a + b = |a + b|.$$

case $a < 0, b < 0$. Then $a + b < 0$ and

$$|a| + |b| = (-a) + (-b) = -(a + b) = |a + b|.$$

case $a < 0, b > 0$. Then $-a > a, b > -b$, and

$$|a| + |b| = (-a) + b > (a + b), |a| + |b| = (-a) + b > (-a) + (-b) = -(a + b),$$

or, equivalently,

$$|a| + |b| > |a + b|.$$

case $a < 0, b > 0$. This is analogous to the previous case, and again

$$|a| + |b| > |a + b|.$$

The conclusion is that the equality $|a| + |b| = |a + b|$ holds if and only if either $a \geq 0, b \geq 0$, or $a \leq 0, b \leq 0$. Thanslating this back in terms of x, y, z, this means either $x \geq y \geq z$, or $x \leq y \leq z$.

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Problem 4. Show that

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$$

for any numbers $x_1, x_2, \ldots, x_n$.

SOLUTION. We will prove this by induction.
base. For $n = 1$ there is nothing to prove, for $n = 2$, this is the usual triangle inequality that we proved in class.
step. Suppose that for some n,

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

For $n + 1$, denote $a := x_1 + x_2 + \cdots + x_n$, $b := x_{n+1}$. Then

$$\begin{aligned}
&|x_1 + x_2 + \cdots + x_n + x_{n+1}| \\
&= |a + b| \\
&\leq |a| + |b| \\
&= |x_1 + x_2 + \cdots + x_n| + |x_{n+1}| \\
&\leq |x_1| + |x_2| + \cdots + |x_n| + |x_{n+1}|
\end{aligned}$$

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Problem 5. Give formulas for two different sequences whose first four terms are 2,4,6,8. In particular, give a formula that generates a sequence whose first five terms are 2,4,6,8,π.

SOLUTION. One obvious choice of a sequence whose first four terms are 2,4,6,8, is the sequence given by the formula $a_n = 2n$ for all $n \in \mathbb{N}$. Its fifth term is 10.

There is more than one way to get a formula for a sequence whose first five terms are 2,4,6,8,π. Here are some possible choices:

$$\begin{aligned} a_n &= 2n + [\![\frac{n}{5}]\!](\pi - 2n), \\ a_n &= 2n + \frac{(n-1)(n-2)(n-3)(n-4)}{24}(\pi - 2n) \end{aligned}$$

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Problem 6. On IQ tests one frequently encounters statements such as "what is the next term in the sequence $3, 1, 4, 1, 5, \ldots$?". In terms of our definition of a sequence is this correct usage? (By the way, what do you suppose the next term in the sequence might be?)
SOLUTION. It is not correct usage. There are infinitely many sequences starting with $3, 1, 4, 1, 5, \ldots$, so talking about *the* sequence does not make sense. The next term could be 1; the first several terms could be:

$$3, 1, 4, 1, 5, 1, 6, 1, 7, 1, 8, 1, 9, 1, 10, 1, 11, 1, 12, 1, \ldots$$

or it could be a 9; the first several terms could be:

$$3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9, 3, 2, 3, 8, 4, 6, 2, 6, 4, 3, \ldots$$

or could be anything else, for that matter.

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Problem 7. Consider the sequence defined recursively by

$$x_1 = \sqrt{2}, \qquad x_n = \sqrt{2 + x_{n-1}}.$$

Show, by induction, that $x_n < x_{n+1}$ forr all n.
SOLUTION.
base:

$$\begin{aligned} 0 &< \sqrt{2} \\ 2 &< 2 + \sqrt{2} \\ \sqrt{2} &< \sqrt{2 + \sqrt{2}} \\ x_1 &< x_2 \end{aligned}$$

step: By the inductional hypothesis, for n,

$$x_n < x_{n+1}.$$

Then,

$$\begin{aligned} 2 + x_n &< 2 + x_{n+1} \\ \sqrt{2 + x_n} &< \sqrt{2 + x_{n+1}} \\ x_{n+1} &< x_{n+2} \end{aligned}$$

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