Dept. of Math. Sci., WPI
MA 3831 Advanced Calculus - I
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Homework Assignment 3

Solutions

Problem 1. The sequence $s_n = (-1)^n$ does not converge. For what values of $\varepsilon > 0$ is it nonetheless true that there is an integer N so that $|s_n - 1| < \varepsilon$ whenever $n \geq N$? For what values of $\varepsilon > 0$ is it nonetheless true that there is an integer N so that $|s_n - 0| < \varepsilon$ whenever $n \geq N$?

SOLUTION. Observe that $|s_n - 1|$ is equal to 0 for n even and to 2 for n odd. Therefore, we will have

$$\exists N \forall n \geq N : |s_n - 1| < \varepsilon$$

if and only if $\varepsilon > 2$.

Similarly, $|s_n - 0| = 1$ for all n, and so we will have

$$\exists N \forall n \geq N : |s_n - 0| < \varepsilon$$

if and only if $\varepsilon > 1$.

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Problem 2. If the sequence s_n is bounded, show that s_n/n is convergent.

SOLUTION. The fact that s_n is bounded means that there exist real numbers, a and b, such that, for all $n \in \mathbb{N}$:

$$a \leq s_n \leq b.$$

Dividing by n, we get

$$\frac{a}{n} \leq \frac{s_n}{n} \leq \frac{b}{n} \qquad \text{for all } n \in \mathbb{N}.$$

Observe that

$$\lim_{n\to\infty} \frac{a}{n} = \lim_{n\to\infty} \frac{b}{n} = 0.$$

By the Squeeze Theorem,

$$\lim_{n\to\infty} \frac{s_n}{n} = 0.$$

In particular, we see that the sequence s_n/n is convergent.

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Problem 3. By immitating the proof of the first part of Theorem 2.15, show that

$$\lim_{n\to\infty}(s_n - t_n) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} t_n.$$

SOLUTION. Denote

$$\lim_{n\to\infty} s_n =: S, \qquad \lim_{n\to\infty} t_n =: T.$$

Let an $\varepsilon > 0$ be given. Choose N_1 so that, whenever $n \geq N_1$, we have

$$|s_n - S| < \frac{\varepsilon}{2}.$$

Choose N_2 so that, whenever $n \geq N_2$, we have

$$|t_n - T| < \frac{\varepsilon}{2}.$$

Put $N := \max\{N_1, N_2\}$. Then, whenever $n \geq N$, we have:

$$|(s_n - t_n) - (S - T)| = |(s_n - S) - (t_n - T)| \leq |s_n - S| + |t_n - T| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

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Problem 4. Prove Theorem 2.16 but verifying and using the inequality

$$|s_n t_n - ST| \leq |(s_n - S)(t_n - T)| + |S(t_n - T)| + |T(s_n - s)|$$

in place of inequality (1). Which proof do you prefer?
SOLUTION. By simple algebra,

$$s_n t_n = ((s_n - S) + S)((t_n - T) + T) = (s_n - S)(t_n - T) + S(t_n - T) + T(s_n - S) + ST,$$

or, equivalently,

$$s_n t_n - ST = (s_n - S)(t_n - T) + S(t_n - T) + T(s_n - S).$$

By the triangle inequality,

$$|s_n t_n - ST| \leq |(s_n - S)(t_n - T)| + |S(t_n - T)| + |T(s_n - s)|.$$

We can now use this inequality to prove Theorem 2.16.

Let an $\varepsilon > 0$ be given. Choose N_1 so that, whenever $n \geq N_1$, we have

$$|s_n - S| < \frac{\varepsilon}{3(|T| + 1) + \varepsilon}.$$

Choose N_2 so that, whenever $n \geq N_2$, we have

$$|t_n - T| < \frac{\varepsilon}{3(|S| + 1) + \varepsilon}.$$

Put $N := \max\{N_1, N_2\}$. Then, whenever $n \geq N$, we have:

$$\begin{aligned}
&|s_n t_n - ST| \\
&\quad \leq \quad |(s_n - S)(t_n - T)| + |S(t_n - T)| + |T(s_n - s)| \\
&\quad < \quad \frac{\varepsilon}{3(|T| + 1) + \varepsilon} \cdot \frac{\varepsilon}{3(|S| + 1) + \varepsilon} + \frac{|S|\varepsilon}{3(|S| + 1) + \varepsilon} + \frac{|T|\varepsilon}{3(|T| + 1) + \varepsilon} \\
&\quad \leq \quad \frac{\varepsilon}{\varepsilon} \cdot \frac{\varepsilon}{3(1)} + \frac{|S|\varepsilon}{3|S|} + \frac{|T|\varepsilon}{3|T|} \\
&\quad = \quad \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&\quad = \quad \varepsilon.
\end{aligned}$$

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Problem 5. A careless student gives the following as a proof of Theorem 2.16. Find the flaw:

"Suppose that $\varepsilon > 0$. Choose N_1 so that

$$|s_n - S| < \frac{\varepsilon}{2|T| + 1}$$

if $n \geq N_1$, and also choose N_2 so that

$$|t_n - T| < \frac{\varepsilon}{2|s_n| + 1}$$

if $n \geq N_2$. If $n \geq \max\{N_1, N_2\}$ then

$$|s_n t_n - ST| \leq |s_n|\,|t_n - T| + |T|\,|s_n - S|$$

$$\leq |s_n| \left(\frac{\varepsilon}{2|s_n|+1}\right). + |T| \left(\frac{\varepsilon}{2|T|+1}\right) < \varepsilon$$

Well, that works!"

SOLUTION. The flaw is in the sentence: "Choose N_2 so that

$$|t_n - T| < \frac{\varepsilon}{2|s_n|+1}$$

if $n \geq N_2$." Since both the lefthand and righthand sides depend on n, it is not obvious why this can be done.

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Problem 6. A careless student gives the following as a proof of the squeeze theorem. Find the flaw:

"If $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = L$, then take limits in the inequality

$$s_n \leq x_n \leq t_n$$

to get $L \leq \lim_{n\to\infty} x_n \leq L$. This can only be true if $\lim_{n\to\infty} x_n = L$. "

SOLUTION. To prove the squeeze theorem, we must show two things: that $\lim_{n\to\infty} x_n$ exists, and that it is equal to L. The student's argument *assumes* that $\lim_{n\to\infty} x_n$ exists, and, under this assumption, shows that the limit must be equal to L. It does not show that the limit exists in the first place.

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Problem 7. Consider the sequence $s_1 = 1$ and $s_n = \frac{2}{s_{n-1}^2}$. We argue that if $s_n \to L$ then $L = \frac{2}{L^2}$, and so $L^3 = 2$ or $L = \sqrt[3]{2}$. Our conclusion is that $\lim_{n\to\infty} s_n = \sqrt[3]{2}$. Do you have any criticisms of this argument?

SOLUTION. Again, the argument validly shows that, *if* the limit exists, it must be equal to $\sqrt[3]{2}$. However, nothing in the argument shows that $\lim_{n\to\infty} s_n$ exists in the first place. (Incidentally, it turns out that the sequence actually diverges.)

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