Dept. of Math. Sci., WPI
MA 3831 Advanced Calculus - I
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Homework Assignment 4

Solutions

Problem 1. Show that $\limsup_{n\to\infty}(-x_n) = -(\liminf_{n\to\infty} x_n)$.
SOLUTION. For every $n \in \mathbb{N}$, we have

$$\sup_{k\geq n}(-x_k) = -\inf_{k\geq n}(x_k).$$

Taking limits, as $n \to \infty$, we get

$$\lim_{n\to\infty}\sup_{k\geq n}(-x_k) = -\lim_{n\to\infty}\inf_{k\geq n}(x_k),$$

which exactly means

$$\limsup_{n\to\infty}(-x_n) = -(\liminf_{n\to\infty} x_n)$$

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Problem 2. Show that for any bounded sequences $a_1, a_2, \ldots, a_n \ldots$ and $b_1, b_2, \ldots, b_n \ldots$ of positive numbers

$$\limsup_{n\to\infty}(a_n b_n) \leq (\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n)$$

SOLUTION. For $n \in \mathbb{N}$, $j \geq n$, we have

$$\begin{aligned} a_j &\leq \sup_{k\geq n} a_k \\ b_j &\leq \sup_{k\geq n} b_k. \end{aligned}$$

Since we are dealing with positive numbers, we can multilpy the two inequalities to get, for all $j \geq n$,

$$a_j b_j \leq \sup_{k\geq n} a_k \sup_{k\geq n} b_k.$$

Since this is true for all $j \geq n$, and the righthand side does not depend on j, we can assert that

$$\sup_{j \geq n}(a_j b_j) \leq (\sup_{k \geq n} a_k)(\sup_{k \geq n} b_k).$$

It remains to let $n \to \infty$.

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Problem 3. If a sequence $a_1, a_2, \ldots, a_n \ldots$ has no convergent subsequences, what can you state about the lim sups and lim infs of the sequence?

SOLUTION. We know that if $\limsup_{n\to\infty} a_n = L \in \mathbb{R}$, then there must be a subsequence converging to L, and similarly for the liminf. Therefore, if the sequence has no converging subsequence, this means that its liminf and limsup cannot be finite real numbers. In other words, one of the following three cases must hold:

$$\liminf_{n\to\infty} a_n = -\infty, \quad \text{and} \quad \limsup_{n\to\infty} a_n = -\infty,$$
$$\liminf_{n\to\infty} a_n = -\infty, \quad \text{and} \quad \limsup_{n\to\infty} a_n = \infty,$$
$$\liminf_{n\to\infty} a_n = \infty, \quad \text{and} \quad \limsup_{n\to\infty} a_n = \infty.$$

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Problem 4. A function f is defined by

$$f(x) = \lim_{n\to\infty} \frac{1}{x^n + x^{-n}}$$

at every value of x for which the limit exists. What is the domain of the function?

SOLUTION. We will cosider several cases for the argumet, x.

case $x = 0$. Then x^{-n} is undefined and f is undefined, too.

case $|x| < 1$. Then f is well defined and is equal to 0.

case $|x| > 1$. This is symmetric to the previous case, because $|x^{-1}| < 1$, and x and x^{-1} appear symmetrically in the formula for f. Again, f is well defined and is equal to 0.

case $x = 1$. The function is well defined and equal to $1/2$.

case $x = -1$. The limit does not exist, f is undefined.
Putting these together, we see that f is undefined only for $x = 0$ and $x = -1$. Thus, the domain of f is

$$\mathbb{R} \setminus \{-1, 0\} = (-\infty, -1) \cup (-1, 0) \cup (0, +\infty).$$

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Problem 5. A sequence $s_1, s_2, \ldots, s_n, \ldots$ is said to be *contractive* if there is a positive number $0 < r < 1$ so that

$$|s_{n+1} - s_n| \le r|s_n - s_{n-1}|$$

for all $n = 2, 3, 4, \ldots$.

(a) Show that the sequence defined by $s_1 = 1$ and $s_n = (4 + s_{n-1})^{-1}$ for $n = 2, 3, \ldots$ is contractive.

(b) Show that every contractive sequence is Cauchy.

(c) Show that a sequence can satisfy the condition

$$|s_{n+1} - s_n| < |s_n - s_{n-1}|$$

for all $n = 2, 3, 4, \ldots$ and not be contractive, nor even convergent.

(d) Is every convergent sequence contractive?

SOLUTION.

(a) It can easily be shown (e.g. by induction) that all terms of the sequence are positive. (With a little more work, it can be shown, also by induction, that $s_n \ge \frac{1}{5}$ for all $n \in \mathbb{N}$.) We have

$$(4 + s_n)(4 + s_{n-1}) \ge (4 + 0)(4 + 0) = 16 \qquad \text{for all } n \ge 2.$$

Therefore, for $n \ge 2$, we have:

$$\begin{aligned} & |s_{n+1} - s_n| \\ &= \left| \frac{1}{4 + s_n} - \frac{1}{4 + s_{n-1}} \right| \\ &= \left| \frac{s_{n-1} - s_n}{(4 + s_n)(4 + s_{n-1})} \right| \\ &\le \frac{|s_n - s_{n-1}|}{16}, \end{aligned}$$

in view of the inequality we proved above. This shows that the given sequence is contractive with $r = 1/16$. (We could have gotten an even smaller r by using the fact that all $s_n \geq 1/5$:

$$(4 + s_n)(4 + s_{n-1}) \geq (4.2)(4.2) = 17.64,$$

which would yield $r = 1/17.64$.)

(b) For a contractive sequence we have:

$$|s_3 - s_2| \leq r|s_2 - s_1|,$$

$$|s_4 - s_3| \leq r|s_3 - s_2| \leq r^2|s_2 - s_1|,$$

$$|s_5 - s_4| \leq r|s_4 - s_3| \leq r^3|s_2 - s_1|,$$

etc., we can show by induction that, for all $n \geq 2$:

$$|s_{n+1} - s_n| \leq r^{n-1}|s_2 - s_1|.$$

Since $0 < r < 1$, we know from Calculus that

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}.$$

and that

$$\lim_{n\to\infty} r^n = 0.$$

We are now prepared to show that the contractive sequence we were given is Cauchy. Let an $\varepsilon > 0$ be given. Choose an integer N so that

$$\frac{|s_2 - s_1|}{1-r} r^{N-1} < \varepsilon.$$

Then, whenever $m > n \geq N$, we have:

$$\begin{aligned}
&|s_m - s_n| \\
&= |(s_m - s_{m-1}) + (s_{m-1} - s_{m-2}) + \cdots + (s_{n+1} - s_n)| \\
&\leq |s_{n+1} - s_n| + \cdots + |s_{m-1} - s_{m-2}| + |s_m - s_{m-1}| \\
&\leq (r^{n-1} + r^n + \cdots + r^{m-3} + r^{m-2})|s_2 - s_1| \\
&\leq r^{n-1}(1 + r + r^2 + \cdots)|s_2 - s_1| \\
&= \frac{|s_2 - s_1|}{1-r} r^{n-1} \\
&\leq \frac{|s_2 - s_1|}{1-r} r^{N-1} \\
&< \varepsilon
\end{aligned}$$

(c) Let $s_n = \sqrt{n}$. This is clearly a sequence that diverges to infinity, so it cannot be Cauchy, and, by part (b), it cannot therefore be contractive. On the other hand, for all $n \in I\!N$,

$$|s_{n+1} - s_n| = |\sqrt{n+1} - \sqrt{n}| = \frac{1}{\sqrt{n^2+n}},$$

and similarly, for $n \geq 2$,

$$|s_n - s_{n-1}| = \frac{1}{\sqrt{n^2-n}}.$$

We see that

$$|s_{n+1} - s_n| < |s_n - s_{n-1}|$$

for all $n = 2, 3, 4, \ldots$.

(d) A sequence can converge without being contractive. A simple example is the sequence $s_n = 1/n$, which we know converges to 0. On the other hand, this sequence is not contractive. Indeed, let $r > 0$ be a real number such that

$$|s_{n+1} - s_n| \leq r|s_n - s_{n-1}| \qquad \text{for all } n \geq 2.$$

Then, for all $n \geq 2$, we must have:

$$r \geq \frac{|s_{n+1} - s_n|}{|s_n - s_{n-1}|} = \frac{n-1}{n+1}.$$

Letting $n \to \infty$ and using the order properties of limits, we see that $r \geq 1$.

Another, even simpler, example of a converging non-contractive sequence is:

$$1, 1/2, 1/2, 1/4, 1/4, 1/8, 1/8, 1/16, 1/16, \ldots.$$

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Problem 6. Determine the set of interior points, accumulation points, isolated points, and boundary points for each of the following sets:

(a) $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$

(b) $\{0\} \cup \{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$

(c) $(0,1) \cup (1,2) \cup (2,3) \cup (3,4) \cup \ldots \cup (n, n+1) \cup \ldots$

(d) $(1/2, 1) \cup (1/4, /2) \cup (1/8, 1/4) \cup (1/16, 1/8) \cup \ldots$

(e) $\{x : |x - \pi| < 1\}$

(f) $\{x : x^2 < 2\}$

(g) $\mathbb{R} \setminus \mathbb{N}$

(h) $\mathbb{R} \setminus \mathbb{Q}$

ANSWER.

(a)
- set of interior points: $\emptyset$.
- set of accumulation points: $\{0\}$.
- set of isolated points: $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$.
- set of boundary points: $\{0\} \cup \{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$.

(b)
- set of interior points: $\emptyset$.
- set of accumulation points: $\{0\}$.
- set of isolated points: $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$.
- set of boundary points: $\{0\} \cup \{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$.

(c)
- set of interior points: $(0,1) \cup (1,2) \cup (2,3) \cup (3,4) \cup \ldots \cup (n, n+1) \cup \ldots$.
- set of accumulation points: $[0, \infty)$.
- set of isolated points: $\emptyset$.
- set of boundary points: $\{0, 1, 2, 3, 4, \ldots\}$

(d)
- set of interior points: $(1/2, 1) \cup (1/4, /2) \cup (1/8, 1/4) \cup (1/16, 1/8) \cup \ldots$.
- set of accumulation points: $[0, 1]$.
- set of isolated points: $\emptyset$.
- set of boundary points: $\{0, 1, 1/2, 1/4, 1/8, 1/16, \ldots\}$.

(e)
- set of interior points: $(\pi - 1, \pi + 1)$

- set of accumulation points: $[\pi - 1, \pi + 1]$
- set of isolated points: $\emptyset$.
- set of boundary points: $\{\pi - 1, \pi + 1\}$

(f)
- set of interior points: $(-\sqrt{2}, \sqrt{2})$.
- set of accumulation points: $[-\sqrt{2}, \sqrt{2}]$.
- set of isolated points: $\emptyset$.
- set of boundary points: $\{-\sqrt{2}, \sqrt{2}\}$.

(g)
- set of interior points: $\mathbb{R} \setminus \mathbb{N}$.
- set of accumulation points: $\mathbb{R}$.
- set of isolated points: $\emptyset$.
- set of boundary points: $\mathbb{N}$.

(h)
- set of interior points: $\emptyset$.
- set of accumulation points: $\mathbb{R}$.
- set of isolated points: $\emptyset$.
- set of boundary points: $\mathbb{R}$.

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Problem 7. Let E be a set and $x_1, x_2, \ldots, x_n, \ldots$ a sequence of elements of E. Suppose that $\lim_{n\to\infty} x_n = x$ and that x is an isolated point of E. Show that there is an integer N so that $x_n = x$ for all $n \geq N$.

SOLUTION. Since x is an isilated poit of E, we can find an $\varepsilon > 0$, so that $(x - \varepsilon, x + \varepsilon) \cap E = \{x\}$. For this $\varepsilon > 0$, find an integer N so that, for all $n \geq N$, we have $|x_n - x| < \varepsilon$ (we can do this because $\lim_{n\to\infty} x_n = x$). Then, for all $n \geq N$,

$$x_n \in (x - \varepsilon, x + \varepsilon) \cap E = \{x\},$$

which means $x_n = x$.

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