Dept. of Math. Sci., WPI
MA 3831 Advanced Calculus - I
Instructor: Bogdan Doytchinov, Term C01

Homework Assignment 1

Due Tuesday, January 23, 2001

Problem 1. Use induction to prove that, for every positive integer n, the number $7^n - 4^n$ is divisible by 3.

Problem 2. Let $P(n)$ denote the statement:

$$1 + 2 + 3 + \cdots + n = \frac{1}{8}(2n+1)^2.$$

(a) Prove that, if $P(n)$ is true for an integer n, then $P(n+1)$ is also true.

(b) Criticize the statement: "By induction, $P(n)$ is true for all n."

(c) Amend $P(n)$ by changing the equality into an inequality that is true for all positive integers n.

Problem 3. For real numbers x, we defined in class $[\![x]\!]$ as the unique integer such that

$$[\![x]\!] \le x < [\![x]\!] + 1.$$

Prove the following properties:

(a) $[\![x+n]\!] = [\![x]\!] + n$ for every integer n.

(b) $[\![-x]\!] = \begin{cases} -[\![x]\!], & \text{if } x \text{ is an integer} \\ -[\![x]\!] - 1, & \text{if } x \text{ is not an integer} \end{cases}$

(c) $[\![x+y]\!]$ is equal to $[\![x]\!] + [\![y]\!]$ or $[\![x]\!] + [\![y]\!] + 1$.

(d) $[\![2x]\!] = [\![x]\!] + [\![x + \frac{1}{2}]\!]$

(e) $[\![3x]\!] = [\![x]\!] + [\![x + \frac{1}{3}]\!] + [\![x + \frac{2}{3}]\!]$

Problem 4. Let $S \subset \mathbb{R}$, $T \subset \mathbb{R}$ be non-empty and bounded above. Prove or disprove:

(a) $\sup(S \cup T) = \max\{\sup S, \sup T\}$

(b) $\sup(S \cap T) = \min\{\sup S, \sup T\}$

Problem 5. Let $x_1, x_2, \ldots, x_n, \ldots$ be a list of *positive* reals. Prove that if the set

$$S = \left\{ z : z = \sum_{k=1}^{n} x_k \text{ for some } n \in \mathbb{N} \right\}$$

is bounded above then there is exactly one number L with the following property:

For each $h > 0$, there are at most finitely many $z \in S$ **not** satisfying the inequality

$$L - h \leq z \leq L.$$