Dept. of Math. Sci., WPI  
MA 3831 Advanced Calculus - I  
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Homework Assignment 1  
Due Tuesday, January 23, 2001

Problem 1. Use induction to prove that, for every positive integer \( n \), the number \( 7^n - 4^n \) is divisible by 3.

Problem 2. Let \( P(n) \) denote the statement:

\[
1 + 2 + 3 + \cdots + n = \frac{1}{8}(2n + 1)^2.
\]

(a) Prove that, if \( P(n) \) is true for an integer \( n \), then \( P(n + 1) \) is also true.

(b) Criticize the statement: “By induction, \( P(n) \) is true for all \( n \).”

(c) Amend \( P(n) \) by changing the equality into an inequality that is true for all positive integers \( n \).

Problem 3. For real numbers \( z \), we defined in class \([z]\) as the unique integer such that

\[
[z] \leq z < [z] + 1.
\]

Prove the following properties:

(a) \([z + n] = [z] + n \) for every integer \( n \).

(b) \([-z] = \begin{cases} -[z], & \text{if } z \text{ is an integer} \\ -[z] - 1, & \text{if } z \text{ is not an integer} \end{cases}\)

(c) \([z + y]\) is equal to \([z] + [y]\) or \([z] + [y] + 1\).

(d) \([2z] = [z] + [z + \frac{1}{2}]\)

(e) \([3z] = [z] + [z + \frac{1}{3}] + [z + \frac{2}{3}]\)
**Problem 4.** Let $S \subset \mathbb{R}$, $T \subset \mathbb{R}$ be non-empty and bounded above. Prove or disprove:

(a) $\sup(S \cup T) = \max\{\sup S, \sup T\}$

(b) $\sup(S \cap T) = \min\{\sup S, \sup T\}$

**Problem 5.** Let $x_1, x_2, \ldots, x_n, \ldots$ be a list of positive reals. Prove that if the set

$$S = \left\{ z : z = \sum_{k=1}^{n} x_k \text{ for some } n \in \mathbb{N} \right\}$$

is bounded above then there is exactly one number $L$ with the following property:

For each $h > 0$, there are at most finitely many $z \in S$ not satisfying the inequality

$$L - h \leq z \leq L.$$