Department of Mathematical Sciences
WPI
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Term C01 SOLUTIONS

MA3831 TEST 1
25 minutes

Problem	Points	Score
1	15	
2	15	
3	15	
4	5	
Total	50	

- This is a 25-minute test. The test has four (4) problems. It is your responsibity to make sure you have all the pages.
- Write your solutions in the space provided below each problem. If you need more space, use the back of the sheet (please indicate clearly if you do so).
- No books or notes are to be consulted.
- Calculators are NOT allowed.
- The solutions you submit must be your own work. You may not look at or copy the work of others.
- Show all your work. No credit will be given for unsupported answers.

GOOD LUCK

1. (15 pts.) Use induction to show that, for every $n \in \mathbb{N}$, the number

$$n^3 + (n+1)^3 + (n+2)^3$$

is divisible by 9.

SOLUTION.
base: for $n = 1$, we have

$$n^3 + (n+1)^3 + (n+2)^3 = 1 + 8 + 27 = 36 = (9)(4),$$

so it is divisible by 9.

step: if the statement is true for n, what can we say for $(n+1)$? By the inductional hypothesis,

$$n^3 + (n+1)^3 + (n+2)^3 = 9k \text{ for some } k \in \mathbb{N}.$$

Then, for $(n+1)$,

$$\begin{aligned} & (n+1)^3 + (n+2)^3 + (n+3)^3 \\ = \; & (n+1)^3 + (n+2)^3 + n^3 + 9n^2 + 27n + 27 \\ = \; & (n^3 + (n+1)^3 + (n+2)^3) + (9n^2 + 27n + 27) \\ = \; & 9k + 9(n^2 + 3n + 3) \\ = \; & 9m, \end{aligned}$$

where $m = k + n^2 + 3n + 3 \in \mathbb{N}$.

2. (15 pts.) We will say that a natural number n is *even*, if $n = 2k$ for some $k \in I\!N$. We will say that a natural number n is *odd*, if $(n + 1)$ is even. Use induction to show that every natural number is either even or odd (in the sense of the above definition).

SOLUTION.

base: for $n = 1$, we have

$$n + 1 = 1 + 1 = 2(1),$$

so 1 is odd.

step: Given that n is either even or odd, we want to show that $(n + 1)$ is even or odd.

case n is odd. This, by the definition given, means $(n + 1)$ is even.

case n is even. Then, $n = 2k$ for some $k \in I\!N$, and for $(n + 1)$, we can write,

$$(n + 1) + 1 = 2k + 2 = 2(k + 1).$$

This means that $(n + 1) + 1$ is even, so $(n + 1)$ is odd.

3. (15 pts.) Let A be a nonempty set of real numbers that is bounded below and let $s = \inf A$. Let

$$B = \{-2x : x \in A\}.$$

Prove that

$$\sup B = -2s.$$

SOLUTION. First, we show that $-2s$ is an upper bound for B. Let y be an arbitrary element of B. Then $y = -2x$ for some $x \in A$. Since s is a lower bound for A,

$$s \leq x$$

and multiplying both sides by -2,

$$-2s \geq -2x = y.$$

Since $y \in B$ was arbitrary, this shows that $-2s$ is indeed an upper bound of B.

Next, we must show that $-2s$ is the *least* of all upper bounds. Let $b < -2s$. Then $-b/2 > s$. Since s is the *greatest* lower bound of A, $-b/2$ cannot be a lower bound of A, and hence there exists an $x \in A$ such that $x < -b/2$. Denote $y = -2x$. Then

$$y = -2x > b,$$

and hence b is not an upper bound of B.

Thus, we see that anything smaller than $-2s$ is not an upper bound of B, while $-2s$ is an upper bound. This exactly means that $-2s$ is the least upper bound of B, i.e.

$$-2s = \sup B.$$

4. (5 pts.) Let E be a nonempty subset of an ordered set (not necessarily of real numbers). Suppose α is a lower bound of E and β is an upper bound of E. Show that $\alpha \leq \beta$.

SOLUTION. Since E is non-empty, we can (and do) choose an element $x \in E$. Since α and β are a lower and upper bound respectively, we have

$$\alpha \leq x, \quad x \leq \beta.$$

By the transitivity property of order,

$$\alpha \leq \beta.$$