Department of Mathematical Sciences
WPI
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Term C01 SOLUTIONS

MA3831 TEST 2
25 minutes

Problem	Points	Score
1	15	
2	15	
3	15	
4	5	
Total	50	

- This is a 25-minute test. The test has four (4) problems. It is your responsibity to make sure you have all the pages.
- Write your solutions in the space provided below each problem. If you need more space, use the back of the sheet (please indicate clearly if you do so).
- No books or notes are to be consulted.
- Calculators are NOT allowed.
- The solutions you submit must be your own work. You may not look at or copy the work of others.
- Show all your work. No credit will be given for unsupported answers.

GOOD LUCK

1. (15 pts.) Let $a_1, a_2, \dots, a_n, \dots$ be a converging sequence, and let

$$A = \lim_{n\to\infty} a_n.$$

Let $M \in \mathbb{N}$ and define $b_n = a_{n+M}$ for all n. Give a precise ε-N proof that the sequence $b_1, b_2, \dots, b_n, \dots$ also converges to A.

SOLUTION. Let an $\varepsilon > 0$ be given. Since the sequence $a_1, a_2, \dots, a_n, \dots$ converges to A, we can find an N such that, whenever $n \geq N$, we have

$$|a_n - A| < \varepsilon.$$

Then, for the same N, whenever $n \geq N$, we also have $n + M \geq N$, and hence

$$|b_n - A| = |a_{n+M} - A| < \varepsilon.$$

2. (15 pts.) Consider the sequence defined recursively by

$$x_1 = \sqrt{2}, \qquad x_n = \sqrt{2 + x_{n-1}}.$$

Show, by induction, that $x_n < x_{n+1}$ for all n.

SOLUTION.
BASE: For $n = 1$, we have:

$$\begin{aligned} 0 &< \sqrt{2} \\ 2 &< 2+\sqrt{2} \\ \sqrt{2} &< \sqrt{2+\sqrt{2}} \\ x_1 &< x_2 \end{aligned}$$

STEP: Given $x_n < x_{n+1}$ for some n,we have:

$$\begin{aligned} 2 + x_n &< 2 + x_{n+1} \\ \sqrt{2+x_n} &< \sqrt{2+x_{n+1}} \\ x_{n+1} &< x_{n+2} \end{aligned}$$

3. (15 pts.) Let $a_1, a_2, \ldots, a_n, \ldots$ be a sequence of *integer* numbers and suppose that, for all $n \in I\!N$, $a_{n+1} \neq a_n$. Give a precise ε-N proof that such a sequence cannot be converging.

SOLUTION. We will prove this by contradiction. Assume the sequence converges, and denote its limit by L. Then, for $\varepsilon = \frac{1}{3}$, we can find an N such that, for all $n \geq N$,

$$|a_n - L| < \varepsilon.$$

Moreover, for $n \geq N$, we also have $(n+1) \geq N$, so that

$$\begin{aligned} |a_n - L| &< \varepsilon \\ |a_{n+1} - L| &< \varepsilon. \end{aligned}$$

On the other hand, since a_n and a_{n+1} are different integers, $|a_{n+1} - a_n|$ is a positive integer, and hence at least 1.

Thus, for $n \geq N$, we have:

$$1 \leq |a_{n+1} - a_n| = |(a_{n+1} - L) + (L - a_n)| \leq |a_{n+1} - L| + |a_n - L| < 2\varepsilon = \frac{2}{3} < 1,$$

which is the desired contradiction.

4. (5 pts.) Give an example of two *diverging* sequences, $a_1, a_2, \ldots, a_n, \ldots$ and $b_1, b_2, \ldots, b_n, \ldots$, such that the sequence $a_1b_1, a_2b_2, \ldots, a_nb_n, \ldots$ converges to a real number.

SOLUTION. For example,

$$a_n = (-1)^n, b_n = (-1)^{n+1} \text{ for all } n.$$