Department of Mathematical Sciences
WPI
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Term C01 SOLUTIONS

MA3831 TEST 3
25 minutes

Problem	Points	Score
1	15	
2	15	
3	15	
4	5	
Total	50	

- This is a 25-minute test. The test has four (4) problems. It is your responsibity to make sure you have all the pages.
- Write your solutions in the space provided below each problem. If you need more space, use the back of the sheet (please indicate clearly if you do so).
- No books or notes are to be consulted.
- Calculators are NOT allowed.
- The solutions you submit must be your own work. You may not look at or copy the work of others.
- Show all your work. No credit will be given for unsupported answers.

GOOD LUCK

1. (15 pts.) If a sequence $x_1, x_2, \ldots, x_n, \ldots$ has the property

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n+1} = L,$$

show that

$$\lim_{n\to\infty} x_n = L.$$

Give a complete ε-N proof.
SOLUTION. Let an $\varepsilon > 0$ be given.
Find an N_1 so that, for every $n \geq N_1$, we have $|x_{2n} - L| < \varepsilon$.
Find an N_2 so that, for every $n \geq N_2$, we have $|x_{2n+1} - L| < \varepsilon$.
Choose an $N \geq \max\{2N_1, 2N_2 + 1\}$. Then, for every $n \geq N$, we have:

case n is even. Then $n = 2k$ for some integer k, and $n \geq N \geq 2N_1$, so $k \geq N_1$, and

$$|x_n - L| = |x_{2k} - L| < \varepsilon.$$

case n is odd. Then $n = 2k + 1$ for some integer k, and $n \geq N \geq 2N_2 + 1$, so $k \geq N_2$, and

$$|x_n - L| = |x_{2k+1} - L| < \varepsilon.$$

We see that, in either case, $|x_n - L| =< \varepsilon$.

2. (15 pts.) Let $a_1, a_2, \ldots, a_n, \ldots$ and $b_1, b_2, \ldots, b_n, \ldots$ be two sequences of real numbers such that

$$\lim_{n\to\infty} |a_n| = A, \qquad \lim_{n\to\infty} |b_n| = B, \qquad \lim_{n\to\infty} |a_n + b_n| = C.$$

Using the properties of limits proved in class, compute

$$\lim_{n\to\infty} |a_n - b_n|.$$

Hint: Look at squares first. $(x+y)^2 + (x-y)^2 = 2x^2 + 2y^2$.

SOLUTION. For every n,

$$|a_n - b_n| = \sqrt{(a_n - b_n)^2} = \sqrt{2a_n^2 + 2b_n^2 - (a_n + b_n)^2}.$$

Then, we let $n \to \infty$. We use the arithmetic properties of limits to conclude that

$$\lim_{n\to\infty} |a_n - b_n| = \sqrt{2(\lim_{n\to\infty} |a_n|)^2 + 2(\lim_{n\to\infty} |b_n|)^2 - (\lim_{n\to\infty} |a_n + b_n|)^2} = \sqrt{2A^2 + 2B^2 - C^2}.$$

3. (15 pts.) Let $\alpha > 1$ and let $s_1, s_2, \ldots, s_n, \ldots$ be a sequence of positive numbers such that $s_{n+1} > \alpha s_n$ for all n. Show that $s_n \to \infty$.

Hint: Show first that the sequence is monotonic, and then that it is unbounded.

SOLUTION. Since all the terms are positive and $\alpha > 1$, we have for all n:

$$s_{n+1} > \alpha s_n \geq s_n,$$

i.e. the sequence is strictly increasing.

Next, we will show (by contradiction) that the sequence is unbounded. Indeed, assume for purposes of controversy that the sequence is bounded. Then, its range must have a finite supremum, L. Since $\alpha > 1$, then $L/\alpha < L$, and hence L/α is not an upper bound for the range of the sequence. In particular, we can find an n such that $s_n > L/\alpha$. Then, we have

$$s_{n+1} > \alpha s_n > \alpha \frac{L}{\alpha} = L,$$

which contradicts the fact that L is an upper bound.

Thus, the given sequence is monotonically increasing and unbounded.

Given any M, we can find an N such that $a_N > M$ (because the sequence is unbounded), and, because of the monotonicity, for all $n \geq N$,

$$a_n \geq a_N > M.$$

4. (5 pts.) Let $s_1, s_2, \ldots, s_n, \ldots$ be a sequence all of whose terms lie inside an interval $[a, b]$. Prove that $\lim_{n\to\infty} s_n/n$ exists. State clearly which theorem(s) you are using.

Solution. This problem is very similar to the homework problem 2.6.2, but is actually problem 2.8.2.

For all n, we have:

$$a \leq s_n \leq b.$$

Dividing by n,

$$\frac{a}{n} \leq \frac{s_n}{n} \leq \frac{b}{n}.$$

Observe that

$$\lim_{n\to\infty} \frac{a}{n} = \lim_{n\to\infty} \frac{b}{n} = 0.$$

By the Squeeze Theorem,

$$\lim_{n\to\infty} \frac{s_n}{n} = 0.$$