Department of Mathematical Sciences
WPI
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Term C01 SOLUTIONS

MA3831 TEST 4
25 minutes

Problem	Points	Score
1	15	
2	15	
3	15	
4	5	
Total	50	

- This is a 25-minute test. The test has four (4) problems. It is your responsibity to make sure you have all the pages.
- Write your solutions in the space provided below each problem. If you need more space, use the back of the sheet (please indicate clearly if you do so).
- No books or notes are to be consulted.
- Calculators are NOT allowed.
- The solutions you submit must be your own work. You may not look at or copy the work of others.
- Show all your work. No credit will be given for unsupported answers.

GOOD LUCK

1. (15 pts.) Show directly that if $s_1, s_2, \ldots, s_n, \ldots$ is a Cauchy sequence, then so too is $|s_1|, |s_2|, \ldots, |s_n|, \ldots$. (Do not use the fact that every Cauchy sequence is convergent.)
SOLUTION. Let an $\varepsilon > 0$ be given. Since the original sequence is Cauchy, we can find an N such that, whenever $m, n \geq N$, we have

$$|s_n - s_m| < \varepsilon.$$

For the very same N, using the triangle inequality, we have:

$$||s_n| - |s_m|| \leq |s_n - s_m| < \varepsilon.$$

2. (15 pts.) Show that for any bounded sequences $a_1, a_2, \ldots, a_n \ldots$ and $b_1, b_2, \ldots, b_n \ldots$ of positive numbers

$$\limsup_{n\to\infty}(a_n b_n) \leq (\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n)$$

SOLUTION. For $n \in I\!N$, $j \geq n$, we have

$$\begin{aligned} a_j &\leq \sup_{k\geq n} a_k \\ b_j &\leq \sup_{k\geq n} b_k. \end{aligned}$$

Since we are dealing with positive numbers, we can multilpy the two inequalities to get, for all $j \geq n$,

$$a_j b_j \leq \sup_{k\geq n} a_k \sup_{k\geq n} b_k.$$

Since this is true for all $j \geq n$, and the righthand side does not depend on j, we can assert that

$$\sup_{j\geq n}(a_j b_j) \leq (\sup_{k\geq n} a_k)(\sup_{k\geq n} b_k).$$

It remains to let $n \to \infty$.

3. (15 pts.) Let E be a set and $x_1, x_2, \ldots, x_n, \ldots$ a sequence of distinct points, not necessarily elements of E. Suppose that $\lim_{n\to\infty} x_n = x$ and that $x_{2n} \in E$ and $x_{2n+1} \notin E$ for all n. Show that x is a boundary point of E.

SOLUTION. Let an $\varepsilon > 0$ be given. We will show that the open iterval $(x - \varepsilon, x + \varepsilon)$ contains both points from E and from its complement.

Indeed, since $\lim_{n\to\infty} x_n = x$, we can find an N, so that for all $n \geq N$,

$$|x_n - x| < \varepsilon,$$

or, equivalently,

$$x_n \in (x - \varepsilon, x + \varepsilon).$$

In particular, for an arbitrary $k \geq N/2 + 1$,

$$x_{2k} \in (x - \varepsilon, x + \varepsilon), \qquad x_{2k} \in E,$$

$$x_{2k+1} \in (x - \varepsilon, x + \varepsilon), \qquad x_{2k+1} \notin E.$$

4. (5 pts.) Express the closed interval $[0, 1]$ as an intersection of a sequence of open sets. Can it also be expressed as the union of a sequence of open sets?

SOLUTION.

$$[0, 1] = \bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1 + \frac{1}{n} \right).$$

It is not possible to express $[0, 1]$ as the union of a sequence of open sets. The union of any collection of open sets is again open, while $[0, 1]$ is not.