Dept. of Math. Sci., WPI
MA 3831 Advanced Calculus - 2
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Homework Assignment 1

Solutions

Problem 1. Which of the following functions defined for pairs of numbers x and y are metrics on $\mathbb{R}$?

(a) $d(x, y) = |x| + |y|$

(b) $d(x, y) = (x - y)^2$

(c) $d(x, y) = \sqrt{|x - y|}$

(d) $d(x, y) = \min\{1, |x - y|\}$

(e) $d(x, y) = \frac{|x-y|}{1+|x-y|}$

(f) $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$.

Solution. To verify that $d : X \times X \to \mathbb{R}$ is a metric, we need to check three conditions:

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

(Remark: from these, it follows that $d(x, y) \geq 0$ for all $x, y \in X$.)
With this in mind, we see, for the given functions:

(a) d is not a metric, (i) is violated: $d(1, 1) = 2 \neq 0$.

(b) d is not a metric, (iii) is violated: for $x = 0$, $y = 1$, $z = 2$:

$$d(x, z) = 4 > 1 + 1 = d(x, y) + d(y, z).$$

(c) d is a metric. The conditions (i) and (ii) are obvious, for (iii) we have:

$$\begin{aligned} d(x,z) &= \sqrt{|x-z|} = \sqrt{|(x-y)+(y-z)|} \leq \sqrt{|x-y| + |x-z|} \\ &\leq \sqrt{|x-y| + 2\sqrt{|x-y|}\sqrt{|y-z|} + |y-z|} \\ &= \sqrt{|x-y|} + \sqrt{|y-z|} \\ &= d(x,y) + d(y,z). \end{aligned}$$

(d) d is a metric - this follows from Problem 3.

(e) d is a metric - this follows from Problem 2.

(f) d is a metric, and this can be easily verified (all three conditions are obvious). It is called *the discrete metric*. A metric space which has the discrete metric is called a *disctrete metric space*.

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Problem 2. Let (X, d) be a metric space. Define a function $e : X \times X \to \mathbb{R}$ by

$$e(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Prove that e is a metric, that $e(x, y) \leq d(x, y)$, and that $e(x, y) \leq 1$ for all $x, y \in X$.

Solution. The inequalities $e(x, y) \leq d(x, y)$, and $e(x, y) \leq 1$ for all $x, y \in X$ are obvious, conditions (i) and (ii) are also immediate from the definition. The only thing that needs proof is (iii), the triangular inequality.

We argue as follows. For $r \geq 0$, denote

$$G(r) = \frac{r}{1 + r}.$$

From Calculus,

$$G'(r) = \frac{1}{(1 + r)^2} > 0, \quad \text{for all } r \geq 0.$$

In other words, the function G is strictly increasing on the interval $[0, \infty)$.

Let $x, y, z \in X$ be given. Then

$$\begin{aligned} e(x, z) &= G(d(x, z)) \\ &\leq G(d(x, y) + d(y, z)) \\ &= \frac{d(x, y) + d(y, z)}{1 + d(x, y) + d(y, z)} \\ &= \frac{d(x, y)}{1 + d(x, y) + d(y, z)} + \frac{d(y, z)}{1 + d(x, y) + d(y, z)} \\ &\leq \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)} \\ &= e(x, y) + e(y, z). \end{aligned}$$

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Problem 3. Let (X, d) be a metric space. Define a function $e : X \times X \to \mathbb{R}$ by

$$e(x, y) = \min\{1, d(x, y)\}.$$

Prove that e is a metric, that $e(x, y) \leq d(x, y)$, and that $e(x, y) \leq 1$ for all $x, y \in X$.

Solution. The inequalities $e(x, y) \leq d(x, y)$, and $e(x, y) \leq 1$ for all $x, y \in X$ are obvious, conditions (i) and (ii) are also immediate from the definition. The only thing that needs proof is (iii), the triangular inequality. Let $x, y, z \in X$ be given. Then

$$\begin{aligned} e(x, z) &= \min\{1, d(x, z)\} \leq \min\{1, d(x, y) + d(y, z)\} \\ &\leq \min\{1, d(x, y)\} + \min\{1, d(y, z)\} \\ &= e(x, y) + e(y, z). \end{aligned}$$

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Problem 4. (Product Spaces) Given two metric spaces we can form a product metric space. Let (X_1, d_1) and (X_2, d_2) be metric spaces. The set

$$X_1 \times X_2 = \{(x_1, x_2) : x_1 \in X_1, x_2 \in X_2\}$$

is called the Cartesian product of X_1 and X_2. For

$$u = (x_1, x_2) \text{ and } v = (y_1, y_2) \text{ in } X_1 \times X_2$$

define

$$d(u, v) = d_1(x_1, y_1) + d_2(x_2, y_2).$$

(a) Prove that d is a metric on $X_1 \times X_2$.

(b) Let $X_1, X_2 = \mathbb{R}$, $d_1 = d_2$ be the usual euclidean metric on $\mathbb{R}$. Calculate $d(u, v)$, where $u = (0, 1)$ and $v = (-3, 4)$ are points in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

Solution.

(a) We check the three conditions:

(i) Since both d_1 and d_2 attain only non-negative values, $d(u,v)=0$ if and only if both $d_1(x_1,y_1)=0$ and $d_2(x_2,y_2)=0$. This happens exactly when $x_1=y_1$ and $x_2=y_2$, i.e., when $u=v$.

(ii) this one is immediate from the formula.

(iii) (triangular inequality) Given $u=(x_1,x_2)$, $v=(y_1,y_2)$, $w=(z_1,z_2)\in X_1\times X_2$,

$$\begin{aligned} d(u,w) &= d_1(x_1,z_1)+d_2(x_2,z_2) \\ &\le d_1(x_1,y_1)+d_1(y_1,z_1)+d_2(x_2,y_2)+d_2(y_2,z_2) \\ &= d(u,v)+d(v,w) \end{aligned}$$

(b) $d(u,v)=|0-(-3)|+|1-4|=6$.

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Problem 5. The book defines $x_0\in X$ as an accumulation point of a set A if every neighborhood of x_0 contains infinitely many points of A. Shows that this definition is equivalent to each of the following:

(a) x_0 is an accumulation point of A if for every $\varepsilon>0$ the set

$$A\cap B(x_0,\varepsilon)\setminus\{x_0\}$$

is nonempty.

(b) x_0 is an accumulation point of A if there is a sequence of points $x_n\in A$ so that $x_n\neq x_0$ and $x_n\to x_0$.

(c) x_0 is an accumulation point of A if $x_0\in\overline{A}$ and x_0 is not an isolated point of A.

(d) x_0 is an accumulation point of A if x_0 is not an interior point of $(X\setminus A)\cup\{x_0\}$.

Solution. Let's denote by (e) the statement of the definition:
(e) x_0 is an accumulation point of A if every neighborhood of x_0 contains infinitely many points of A.
The proof of the equivalence will proceed along the following scheme:

$$\text{(e)}\Rightarrow\text{(a)}\Rightarrow\text{(b)}\Rightarrow\text{(e)},\quad \text{(a)}\Rightarrow\text{(c)}\Rightarrow\text{(a)},\quad \text{(a)}\Rightarrow\text{(d)}\Rightarrow\text{(a)}.$$

(e)⇒(a) Let an $\varepsilon > 0$ be given. $B(x_0, \varepsilon)$ is a neighborhood of x_0, and in it we can find infinitely many points of A. In particular, we can find a point $y \in A$ such that $y \neq x_0$. Then $y \in A \cap B(x_0, \varepsilon) \setminus \{x_0\}$, and hence the set $A \cap B(x_0, \varepsilon) \setminus \{x_0\}$ is non-empty.

(a)⇒(b) For every $n \in \mathbb{N}$, we can find a point

$$x_n \in A \cap B\left(x_0, \frac{1}{n}\right) \setminus \{x_0\}.$$

Then $x_1, x_2, \ldots, x_n, \ldots$ is the desired sequence.

(b)⇒(e) Let G be a neighborhood of x_0, and let $x_1, x_2, \ldots, x_n, \ldots$ be a sequence as described in (b). Since $x_n \to x_0$, we can find an N so that $x_n \in G$ for all $n \geq N$. Then the set

$$\{x_n : n \geq N\}$$

is an infinite set of points of A and is also a subset of G.

(a)⇒(c) For every $\varepsilon > 0$, the open ball $B(x_0, \varepsilon)$ contains points from A. Hence, x_0 cannot be an isolated point. Moreover, from the definition of closure, it is immediate that $x_0 \in \overline{A}$.

(c)⇒(a) By the definition of closure, $\overline{A} = A \cup A'$. If $x_0 \in \overline{A}$, but $x_0 \notin A$, then x_0 is an accumulation point in the sense of (a).

If, on the other hand, $x_0 \in A$, then, we use the fact that it is not an isolated point. Then, for every $\varepsilon > 0$, the open ball $B(x_0, \varepsilon)$ contains points from A, different from x_0. Thus,

$$B(x_0, \varepsilon) \cap A \setminus \{x_0\} \neq \emptyset,$$

which is exactly the condition in (a).

(a)⇒(d) Let's reason by contradiction. Suppose, for purposes of controversy, that x_0 *is* an interior point of $(X \setminus A) \cup \{x_0\}$. Then we can find an $\varepsilon > 0$ so that $B(x_0, \varepsilon) \subset (X \setminus A) \cup \{x_0\}$. This is equivalent to saying that

$$B(x_0, \varepsilon) \cap A \setminus \{x_0\} = \emptyset,$$

which yields the desired contradiction.

(d)⇒(a) Along with the reasoning in the previous paragraph, x_0 not being an interior point means that

$$B(x_0, \varepsilon) \cap A \setminus \{x_0\} \neq \emptyset,$$

which is exactly the condition in (a).

Problem 6. Show, in a general metric space, that the open ball is open and that the closed ball is closed. Give an example of a metric space in which a closed ball $B[x, \varepsilon]$ is not necessarily the closure of the open ball $B(x, \varepsilon)$.
Solution. Let (X, d) be a general metric space, and let $x \in X$, $\varepsilon > 0$ be given.

First we will show that the open ball $B(x, \varepsilon)$ is open. To this end, we must show that every point $y \in B(x, \varepsilon)$ is an interior point of $B(x, \varepsilon)$. In other words, we must produce a $\delta > 0$ such that $B(y, \delta) \subset B(x, \varepsilon)$. Here is how we do it.

Since $y \in B(x, \varepsilon)$, we have $d(x, y) < \varepsilon$. Choose $\delta = \varepsilon - d(x, y)$. Then, for every $z \in B(y, \delta)$, we have

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \delta = \varepsilon,$$

and hence $z \in B(x, \varepsilon)$. This exactly means that $B(y, \delta) \subset B(x, \varepsilon)$.

Next, we show that the closed ball $B[x, \varepsilon]$ is closed. Since a set is closed if and only if its complement is open, it will be enough to show that its complement, $X \setminus B[x, \varepsilon]$, is open. To this end, we must show that every point $y \in X \setminus B[x, \varepsilon]$ is an interior point of $X \setminus B[x, \varepsilon]$. In other words, we must produce a $\delta > 0$ such that $B(y, \delta) \subset X \setminus B[x, \varepsilon]$. Here is how we do it.

Since $y \in X \setminus B[x, \varepsilon]$, we have $d(x, y) > \varepsilon$. Choose $\delta = d(x, y) - \varepsilon$. Then, for every $z \in B(y, \delta)$, we have

$$d(x, y) \leq d(x, z) + d(z, y),$$

so

$$d(x, z) \geq d(x, y) - d(z, y) > d(x, y) - \delta = \varepsilon,$$

and hence $z \in X \setminus B[x, \varepsilon]$. This exactly means that $B(y, \delta) \subset X \setminus B[x, \varepsilon]$.

Finally, we note that a closed ball $B[x, \varepsilon]$ is not necessarily the closure of the open ball $B(x, \varepsilon)$. For example, let $X = \mathbb{N}$ with the usual real metric. Then $B(1, 1) = \{1\}$, and $\overline{B(1, 1)} = \{1\}$, but $B[1, 1] = \{1, 2\}$.

Problem 7. Let X denote the set of points

$$\{0\} \cup \{1/k : k = 1, 2, 3, \ldots\}$$

in $\mathbb{R}$ furnished with the usual real metric. Answer the following questions:

(a) Which points are isolated in X?

(b) Which sets are open and which sets are closed?

(c) Which sets are both open and closed?

(d) Is X bounded?

(e) Which sets have a nonempty boundary?

(f) Does X have any accumulation points?

(g) Describe all dense subsets of X.

(h) Is the closure of the open ball $B(x, \varepsilon)$ necessarily the closed ball $B[x, \varepsilon]$?

Solution.

(a) All the points of the form $1/k$ for $k \in \mathbb{N}$.

(b) There are two types of closed sets. First, all finite sets are closed. Second, all infinite sets containing the point 0 are closed.

The open sets are the complements of the closed sets. Thus, they too can be classified into two types. First all cofinite sets (i.e. sets with finite complements), and second, all the sets that do not contain the point 0.

(c) Two types of sets are both open and closed: first, all finite sets that do not contain the point 0, and, second, all the cofinite sets that do contain 0.

(d) Yes, it is. For all $x, y \in X$, $|x - y| \leq 1$.

(e) Observe that each point of the form $1/k$ for some $k \in \mathbb{N}$ is an open set. Thus, the only point in X that can possibly be a boundary point for a set, is the point 0.

A set has a non-empty boundary exactly in the following two cases:

- it contains 0 and its complement is infinite.
- it is infinite and it does not contain 0.

(f) Yes, the point 0 is an accumulation point. All other points are not accumulation points (because they are isolated).

(g) There are exactly two dense subsets of X: first, X itself, and second, the set $\{1/k : k = 1, 2, 3, \ldots\}$.

(h) No. $B(1, 1/2) = \{1\}$, and $\overline{B(1, 1/2)} = \{1\}$, but $B[1, 1/2] = \{1, 1/2\}$.