**Dept. of Math. Sci., WPI**
**MA 3831 Advanced Calculus - 2**
**Instructor: Bogdan Doytchinov, Term D01**

# Homework Assignment 2

Solutions

**Problem 1.** Let $f$ be defined on the set containing the points

$$0, 1, 1/2, 1/4, \ldots, 1/2^n, \ldots$$

only. What values can you assign at these points that will make this function continuous everywhere where it is defined?
**Solution.** Since the points

$$1, 1/2, 1/4, \ldots, 1/2^n, \ldots$$

are isolated, $f$ will be continuos at these points, no matter how we define it. The only accumulation poit is 0. Thus, the function will be continuous everywhere where it is defined, if and only if it is continuous at 0. The necessary and sufficient condition for this is

$$\lim_{n\to\infty} f(1/n) = f(0).$$

Conclusion: The values

$$f(1), f(1/2), \ldots, f(1/2^n), \ldots$$

should form a converging sequence, and $f(0)$ should equal the limit of this sequence.

∎

**Problem 2.** A function $f : [a, b] \to I\!R$ is said to be *Lipshitz* if there is a positive number $M$ so that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in [a, b]$. Show that a Lipshitz function must be continuous. Is the converse true?
**Soultion.** Given $x_0 \in [a, b]$ and $\varepsilon > 0$, choose $\delta := \varepsilon/M$. Then, whenever $x \in [a, b] \cap (x_0 - \delta, x_0 + \delta)$, we have

$$|f(x) - f(x_0)| \leq M|x - x_0| < M\delta = \varepsilon.$$

This exactly means that $f$ is continuous on $[a, b]$.

The converse is not true. For example, the function $f : [-1, 1] \to \mathbb{R}$, given by $f(x) = \sqrt[3]{x}$, is continuous on $[-1, 1]$ but does not satisfy the Lipshitz condition. For $x \neq y$ we have:

$$|f(x) - f(y)| = |\sqrt[3]{x} - \sqrt[3]{y}| = \frac{|x - y|}{\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}}.$$

There is no such constant $M$ that

$$\frac{1}{\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}} \leq M$$

for all $x, y \in [-1, 1]$; by taking $x$ and $y$ close to 0, we can make this expression arbitrarily large.

∎

**Problem 3.** Let $(X, d)$ be a discrete space.

**(a)** What functions $f : X \to \mathbb{R}$ are continuous everywhere?

**(b)** What functions $f : \mathbb{R} \to X$ are continuous everywhere?

**Soluiton.** Recall that a mapping is continuous everywhere if and only if the pre-image of every open set is open, or, equvalently,if the pre-image of every closed set is closed. Also, recall that in a discrete metric space all sets are both open and closed. In view of this, we immediately see that

**(a)** All functions $f : X \to \mathbb{R}$ are continuous everywhere.

**(b)** A function $f : \mathbb{R} \to X$ is continuous everywhere if and only if it is constant. Indeed, let $x_0 := f(0)$. Then the set $f^{-1}(\{x_0\})$ is a subset of $\mathbb{R}$ which is open, closed and nonempty. Thus

$$f^{-1}(\{x_0\}) = \mathbb{R},$$

or, in other fords, $f(t) = f(0)$ for all $t \in \mathbb{R}$.

∎

**Problem 4.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x_1, x_2) = \frac{x_1^2 x_2}{x_1^4 + x_2^2} \quad f(0,0) = 0.$$

Show that $\lim_{x\to 0} f(x, mx) = 0$ for every $m \in \mathbb{R}$, but $f$ is discontinuous at $(0,0)$.

**Solution.** For every $m \in \mathbb{R}$, we have

$$\lim_{x\to 0} f(x, mx) = \lim_{x\to 0} \frac{x^2 mx}{x^4 + m^2 x^2} = \lim_{x\to 0} \frac{mx}{x^2 + m^2} = 0.$$

On the other hand, for the sequence

$$(x_1^{(n)}, x_2^{(n)}) := \left(\frac{1}{n}, \frac{1}{n^2}\right), \quad n = 1, 2, \ldots,$$

we have

$$\lim_{n\to\infty} (x_1^{(n)}, x_2^{(n)}) = (0,0)$$

but

$$\lim_{n\to\infty} f(x_1^{(n)}, x_2^{(n)}) = \frac{1}{2} \neq f(0,0).$$

∎

**Problem 5.** Let $(X, d)$ be a metric space. Prove that $d$ is continuous on $X \times X$, where $X \times X$ is furnished with the product metric.

**Solution.** Let $\rho$ denote the product metric on $X \times X$, i.e., for $(x_1, x_2) \in X \times X$, and $(y_1, y_2) \in X \times X$,

$$\rho((x_1, x_2), (y_1, y_2)) = d(x_1, y_1) + d(x_2, y_2).$$

We will show that $d$ is *uniformly* continuous on $X \times X$.

Indeed, let an $\varepsilon > 0$ be given, and choose $\delta = \varepsilon$. Then, whenever $(x_1, x_2) \in X \times X$ and $(y_1, y_2) \in X \times X$ are such that $\rho((x_1, x_2), (y_1, y_2)) < \delta$, we have

$$\begin{aligned} |d(x_1, x_2) - d(y_1, y_2)| &= |d(x_1, x_2) - d(y_1, x_2) + d(y_1, x_2) - d(y_1, y_2)| \\ &\leq |d(x_1, x_2) - d(y_1, x_2)| + |d(y_1, x_2) - d(y_1, y_2)| \\ &\leq d(x_1, y_1) + d(x_2, y_2) \\ &= \rho((x_1, x_2), (y_1, y_2)) < \delta = \varepsilon. \end{aligned}$$

∎

**Problem 6.** Let $(X, d)$ be a metric space and let $A$ be a nonempty subset of $X$. Define $f : X \to I\!R$ by

$$f(x) = \text{dist}(x, A) = \inf\{d(x, y) : y \in A\}.$$

**(a)** Show that $|f(x) - f(y)| \leq d(x, y)$ for all $x, y \in X$.

**(b)** Show that $f$ defines a continuous real-valued function on $X$.

**(c)** Show that $\{x \in X : f(x) = 0\} = \overline{A}$.

**(d)** Show that $\{x \in X : f(x) > 0\} = \text{int}(X \setminus A)$.

**(e)** Show that, unless $X$ conains only a single point, there exists a continuous real-valued function defined on $X$ that is not constant.

**(f)** If $E \subset X$ is closed and $x_0 \notin E$, show that there is a continuous real-valued function $g$ on $X$ so that $g(x_0) = 1$ and $g(x) = 0$ for all $x \in E$.

**(g)** If $E$ and $F$ are disjoint closed subsets of $X$, show that there is a continuous real-valued function $g$ on $X$ so that $g(x) = 1$ for all $x \in F$ and $g(x) = 0$ for all $x \in E$.

**(h)** If $E$ and $F$ are disjoint closed subsets of $X$, show that there are disjoint open sets $G_1$ and $G_2$ so that $E \subset G_1$ and $F \subset G_2$.

**(i)** In the special case where $X$ is the real line with the usual metric and $K$ denotes the Cantor ternary set, sketch the graph of the function $f(x) = \text{dist}(x, K)$.

**(j)** Give an example of a metric space, a point $x_0$, and a set $A \subset X$ so that $\text{dist}(x_0, A) = 1$ but so that $d(x, x_0) \neq 1$ for every $x \in \overline{A}$.

**Solution.**

**(a)** Let $x, y \in X$ be given. For all $z \in A$, we have:

$$f(x) = \text{dist}(x, A) = \inf\{d(x, \zeta) : \zeta \in A\} \leq d(x, z) \leq d(x, y) + d(y, z).$$

Taking an infimum over $z \in A$, we get

$$f(x) \leq d(x, y) + f(y).$$

By symmetry,

$$f(y) \leq d(x, y) + f(x).$$

Combining these two, we get

$$|f(x) - f(y)| \leq d(x, y).$$

**(b)** From (a), we see immediately that $f$ is actually *uniformly* continuos (and even Lipshitz continuous) on $X$: given $\varepsilon > 0$, we can choose $\delta = \varepsilon$. Then, whenever $d(x, y) < \delta$, we have $|f(x) - f(y)| < \varepsilon$.

**(c)** First of all, observe that $\{x \in X : f(x) = 0\} = f^{-1}(\{0\})$. Since $f$ is continuous, $\{x \in X : f(x) = 0\}$ is closed. Next, clearly, for $x \in A$, $f(x) = 0$. Thus, $\{x \in X : f(x) = 0\}$ is a closed set that includes $A$. Since $\overline{A}$ is the *smallest* closed set that includes $A$, we see that

$$\{x \in X : f(x) = 0\} \supset \overline{A}.$$

To show the equality, we must now check that, converslely, every point of $\overline{A}$ is also in $\{x \in X : f(x) = 0\}$. Let an $x \in \overline{A}$ be given. Then there exists a sequence $x_n \to x$, with $x_n \in A$ for all $n$. Since $f$ is continuous, we have:

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0.$$

**(d)** For this, we will use the result of (c), as well as the properties of pre-images. Since $f$ attains only non-negative values,

$$f^{-1}((-\infty, 0]) = f^{-1}(\{0\}) = \overline{A}$$

and hence,

$$\begin{aligned} \{x \in X : f(x) > 0\} &= f^{-1}((0, \infty)) = f^{-1}(\mathbb{R} \setminus (-\infty, 0]) \\ &= X \setminus f^{-1}(\{0\}) = X \setminus \overline{A} = \int (X \setminus A). \end{aligned}$$

**(e)** Let $x, y \in X$ and $x \neq y$. Consider the function

$$h(z) = d(z, x) = \operatorname{dist}(z, \{x\}).$$

This is a continuous function, and $h(y) = d(x, y) > 0 = h(x)$, so it is not a constant function.

**(f)** Let

$$f(x) = \operatorname{dist}(x, E).$$

Since $E$ is closed, and $x_0 \notin E$, (d) ensures that $f(x_0) > 0$, while $f(x) = 0$ exactly when $x \in E$. Put

$$g(x) := \frac{f(x)}{f(x_0)}.$$

**(g)** Put

$$g(x) := \frac{\operatorname{dist}(x, E)}{\operatorname{dist}(x, E) + \operatorname{dist}(x, F)}.$$

Since $E$ and $F$ are disjoint closed subsets of $X$, the denominator is never 0, so $g$ is well-defined and continuous. It is straightforward to verify that $g(x) = 1$ for all $x \in F$ and $g(x) = 0$ for all $x \in E$.

**(h)** Let $g$ be the function introduced in $g$. Take

$$G_1 := f^{-1}((-\infty, 1/2)), \qquad , G_2 := f^{-1}((1/2, \infty)).$$

**(i)** In the special case where $X$ is the real line with the usual metric and $K$ denotes the Cantor ternary set, sketch the graph of the function $f(x) = \operatorname{dist}(x, K)$ - we did this in class.

**(j)** Take $X = \mathbb{R}$, but with the following metric:

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1 + |x - y|, & \text{if } x \neq y. \end{cases}$$

Take $x_0 = 0$, $A = X \setminus \{0\}$. Then $A$ is closed (in this metric) and $\operatorname{dist}(x_0, A) = 1$, while $d(x, x_0) > 1$ for every $x \in A = \overline{A}$.

∎