Dept. of Math. Sci., WPI
MA 3832 Advanced Calculus - 2
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Homework Assignment 3

Soultions

Problem 1. Consider the set $C[0,1]$ of continuous functions on $[0,1]$ with the two metrics:

$$\begin{aligned} d_1(x,y) &= \max_{0\le t\le 1} |x(t)-y(t)| \\ d_2(x,y) &= \int_0^1 |x(t)-y(t)|\,dt. \end{aligned}$$

Let B_1 be an open ball in $(C[0,1],d_1)$ and let B_2 be an open ball in $C[0,1],d_2)$.

(a) Is B_1 open in $(C[0,1],d_2)$?

(b) Is B_2 open in $(C[0,1],d_1)$?

SOLUTION.

(a) B_1 is *not* open in $(C[0,1],d_2)$. We can reason by contradiction. Suppose, for purposes of controversy, that B_1 is open in the metric d_2, and let $y\in B_1$. Then, for every sequence $x_n \in C[0,1]$ for which $d_2(x_n,y)\to 0$, there must exist an N such that, for all $n\ge N$, $x_n\in B_1$. Consider now the sequence of functions $x_n\in C[0,1]$ given by the formula

$$x_n(t) = \begin{cases} y(t)+\sqrt{n}\,(1-|2nx-1|), & \text{for } 0\le x\le 1/n, \\ y(t), & \text{for } x>1/n. \end{cases}$$

Then $d_2(x_n,y)=1/(2\sqrt{n})\to 0$. On the other hand, $d_1(x_n,y)\to\infty$, so B_1 can contain at most finitely many of the x_n's.

(b) B_2 *is* open in $(C[0,1],d_1)$. Indeed, observe that, for any two functions $x,y\in C[0,1]$,

$$\begin{aligned} d_2(x,y) &= \int_0^1 |x(t)-y(t)|\,dt \\ &\le \int_0^1 \max_{0\le s\le 1} |x(s)-y(s)|\,dt = \int_0^1 d_1(x,y)dt = d_1(x,y). \end{aligned}$$

This shows that the identity mapping $\mathrm{id} : (C[0,1], d_1) \to (C[0,1], d_2)$ is Lipschitz continuous (and, in particular, continuous). B_2 is open in $(C[0,1], d_2)$ (given), therefore, its pre-image $B_2 = \mathrm{id}^{-1}(B_2)$ is open in $(C[0,1], d_1)$.

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Problem 2. Let $C^1[a,b]$ consist of the continuously differentiable functions on $[a,b]$. Define for $f, g \in C^1[a,b]$

$$d(f,g) = \max_{a \le t \le b} |f(t) - g(t)| + \max_{a \le t \le b} |f'(t) - g(t')|.$$

(a) Prove that d is a metric.

(b) Let $D : C^1[a,b] \to C[a,b]$ be defined by $D(f) = f'$. Prove that D is continuous. (Here, as usual, $C[a,b]$ has the sup metric).

SOLUTION.

(a) The only non-trivial property to check is the triangle inequality, the other properties of metric are obviously satisfied. Let $f, g, h \in C^1[a,b]$ be given. In class, we proved the triangle inequality for the sup-norm:

$$\max_{a \le t \le b} |f(t) - h(t)| \le \max_{a \le t \le b} |f(t) - g(t)| + \max_{a \le t \le b} |g(t) - h(t)|.$$

This, applied to the derivatives, gives also

$$\max_{a \le t \le b} |f'(t) - h'(t)| \le \max_{a \le t \le b} |f'(t) - g'(t)| + \max_{a \le t \le b} |g'(t) - h'(t)|.$$

Adding the two inequalities, we get

$$d(f,h) \le d(f,g) + d(g,h).$$

(b) For this part of the problem, let's denote

$$\rho(f,g) = \max_{a \le t \le b} |f(t) - g(t)|, \qquad \text{for } f, g \in C[a,b].$$

Then, for every two functions $f, g \in C^1[a,b]$, we have

$$\begin{aligned} \rho(D(f), D(g)) &= \max_{a \le t \le b} |f'(t) - g'(t)| \\ &\le \max_{a \le t \le b} |f(t) - g(t)| + \max_{a \le t \le b} |f'(t) - g(t')| = d(f,g). \end{aligned}$$

This shows that $D : C^1[a,b] \to C[a,b]$ is Lipschitz continuous (and therefore, continuous).

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Problem 3. Let $C[0,1]$ consist of the continuous functions on $[0,1]$ and furnished with the metric

$$d(f,g) = \int_0^1 |f(t) - g(t)|\, dt.$$

Define $T : C[0,1] \to I\!R$ by

$$T(f) = \int_0^1 f(t)\, dt.$$

Is T continuos?

SOLUTION. Yes, T is continuous, and even Lipschitz continuous with a Lipschitz constant 1. For every $f, g \in C[0,1]$,

$$\begin{aligned} |T(f) - T(g)| &= |\int_0^1 f(t)\, dt - \int_0^1 g(t)\, dt| = |\int_0^1 (f(t) - g(t))\, dt| \\ &\le \int_0^1 |f(t) - g(t)|\, dt = d(f,g). \end{aligned}$$

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Problem 4. Let a function $f : I\!R \to I\!R$ be defined by setting $f(1/n) = c_n$ for $n = 1, 2, 3, \ldots$ where $c_1, c_2, \ldots$ is a given sequence, and elswhere $f(x) = 0$. Find a condition on that sequence so that $f'(0)$ exists.

SOLUTION. By the definition of derivative, $f'(0)$ is defined as the following limit:

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x},$$

which must exist.

By the definition of f,

$$\frac{f(x)}{x} = \begin{cases} nc_n, & \text{if } x = 1/n \text{ for some } n = 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

In every neighborhood of 0 there are infinitely many x's for which $f(x) = 0$. This meams that, if $\lim_{x\to 0} \frac{f(x)}{x}$ exists, it must be 0. On the other hand, if this limit exist, we must also have

$$\lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{f(1/n)}{1/n} = \lim_{n\to\infty} nc_n.$$

We see that $f'(0)$ exists if and only if

$$\lim_{n\to\infty} nc_n = 0,$$

and if this is the case, $f'(0) = 0$.

The necessity of this condition was shown above. The fact that it is also sufficient, can be seen from the following:

Let an $\varepsilon > 0$ be given, and let N be such that for all $n \geq N$ we have $|nc_n| < \varepsilon$. Choose $\delta = 1/N$. Then, whenever $0 < |x| < \delta$, we have either $f(x)/x = 0$ or $f(x)/x = nc_n$ for some $n > N$. In either case,

$$\left|\frac{f(x) - f(0)}{x - 0} - 0\right| = \left|\frac{f(x)}{x}\right| < \varepsilon.$$

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Problem 5. Suppose f satisfies the hypotheses of the mean value theorem on $[a, b]$. Let S be the set of all slopes of chords determined by pairs of points on the graph of f and let

$$D = \{f'(x) : x \in (a, b)\}.$$

(a) Prove that $S \subset D$.

(b) Give an example to show that D can contain numbers not in S.

SOLUTION.

(a) Let $s \in S$. Then there exist $x, y \in [a, b]$ so that $x < y$ and

$$s = \frac{f(x) - f(y)}{x - y}.$$

By the mean value theorem, there exists a $\xi \in (x, y) \subset [a, b]$ such that

$$f'(\xi) = \frac{f(x) - f(y)}{x - y}.$$

Thus,

$$s = f'(\xi) \in D.$$

Since $s \in S$ was arbitrary, we see that $S \subset D$.

(b) Let $[a, b] = [-1, 1]$, and take $f(x) = x^3$. Then, whenever $x, y \in [a, b]$ and $x < y$, we have $f(x) < f(y)$, so

$$\frac{f(x) - f(y)}{x - y} > 0,$$

in particular, $0 \notin S$. On the other hand, $0 = f'(0) \in D$.

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Problem 6. Suppose f is continuous on $[a, b]$ and differentiable on (a, b). If

$$\lim_{x \to a+} f'(x) = C$$

what can you conclude about the right-hand derivative of f at a?

SOLUTION. We can conclude that $f'(a+)$ exists and is equal to C. We actually did this in class, in order to show that a derivative cannot have jump discontinuities. Here is how the argument went:

Let an $\varepsilon > 0$ be given, and let $\delta > 0$ be such that, whenever $a < x < a+\delta$, we have

$$|f'(x) - C| < \varepsilon.$$

Then, for the very same δ, whenever $a < x < a + \delta$, we can find a $\xi \in (a, x)$ by the mean value theorem, so that

$$f'(\xi) = \frac{f(x) - f(y)}{x - y},$$

and therefore

$$\left|\frac{f(x) - f(a)}{x - a} - C\right| = |f'(xi) - C| < \varepsilon,$$

since $a < \xi < x < a + \delta$. This proves that

$$\lim_{x \to a+} \frac{f(x) - f(a)}{x - a} = C.$$

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Problem 7. Let $f : [0, 1] \to \mathbb{R}$ be a differentiable function such that $|f'(x)| \leq M$ for all $x \in (0, 1)$. Show that

$$\left| \int_0^1 f(x)\, dx - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \right| \leq \frac{M}{n}.$$

Solution. By the mean value theorem, for every $x, y \in [0, 1]$, there exists a ξ such that

$$f(x) - f(y) = f'(\xi)(x - y),$$

and hence

$$|f(x) - f(y)| = |f'(\xi)| \cdot |x - y| \leq M|x - y|.$$

We have:

$$\left| \int_0^1 f(x)\, dx - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \right| = \left| \sum_{k=1}^{n} \int_{k/(n-1)}^{k/n} \left(f(x) - \left(\frac{k}{n}\right)\right) dx \right|$$

$$\leq \sum_{k=1}^{n} \int_{k/(n-1)}^{k/n} \left| f(x) - \left(\frac{k}{n}\right) \right| dx \leq \sum_{k=1}^{n} \int_{k/(n-1)}^{k/n} M \left(\frac{k}{n} - x\right) dx$$

$$= M \sum_{k=1}^{n} \int_{k/(n-1)}^{k/n} \left(\frac{k}{n} - x\right) dx = M \sum_{k=1}^{n} \frac{1}{2n^2} = \frac{M}{2n}.$$

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