Problem 1. Consider the set $C[0, 1]$ of continuous functions on $[0, 1]$ with the two metrics:

$$d_1(x, y) = \max_{0 \leq t \leq 1} |x(t) - y(t)|$$

$$d_2(x, y) = \int_0^1 |x(t) - y(t)| \, dt.$$

Let $B_1$ be an open ball in $(C[0, 1], d_1)$ and let $B_2$ be an open ball in $C[0, 1], d_2)$.

(a) Is $B_1$ open in $(C[0, 1], d_2)$?

(b) Is $B_2$ open in $(C[0, 1], d_1)$?

Solution.

(a) $B_1$ is not open in $(C[0, 1], d_2)$. We can reason by contradiction. Suppose, for purposes of controversy, that $B_1$ is open in the metric $d_2$, and let $y \in B_1$. Then, for every sequence $x_n \in C[0, 1]$ for which $d_2(x_n, y) \to 0$, there must exist an $N$ such that, for all $n \geq N$, $x_n \in B_1$. Consider now the sequence of functions $x_n \in C[0, 1]$ given by the formula

$$x_n(t) = \begin{cases} y(t) + \sqrt{n} (1 - |2nx - 1|), & \text{for } 0 \leq x \leq 1/n, \\ y(t), & \text{for } x > 1/n. \end{cases}$$

Then $d_2(x_n, y) = 1/(2\sqrt{n}) \to 0$. On the other hand, $d_1(x_n, y) \to \infty$, so $B_1$ can contain at most finitely many of the $x_n$'s.

(b) $B_2$ is open in $(C[0, 1], d_1)$. Indeed, observe that, for any two functions $x, y \in C[0, 1]$,

$$d_2(x, y) = \int_0^1 |x(t) - y(t)| \, dt$$

$$\leq \int_0^1 \max_{0 \leq s \leq 1} |x(s) - y(s)| \, dt = \int_0^1 d_1(x, y) \, dt = d_1(x, y).$$
This shows that the identity mapping \( \text{id} : (C[0, 1], d_1) \to (C[0, 1], d_2) \) is Lipschitz continuous (and, in particular, continuous). \( B_2 \) is open in \( (C[0, 1], d_2) \) (given), therefore, its pre-image \( B_2 = \text{id}^{-1}(B_2) \) is open in \( (C[0, 1], d_1) \).

\[\]  

**Problem 2.** Let \( C^1[a, b] \) consist of the continuously differentiable functions on \([a, b]\). Define for \( f, g \in C^1[a, b] \)

\[
d(f, g) = \max_{a \leq t \leq b} |f(t) - g(t)| + \max_{a \leq t \leq b} |f'(t) - g'(t)|.
\]

(a) Prove that \( d \) is a metric.

(b) Let \( D : C^1[a, b] \to C[a, b] \) be defined by \( D(f) = f' \). Prove that \( D \) is continuous. (Here, as usual, \( C[a, b] \) has the sup metric).

**Solution.**

(a) The only non-trivial property to check is the triangle inequality, the other properties of metric are obviously satisfied. Let \( f, g, h \in C^1[a, b] \) be given. In class, we proved the triangle inequality for the sup-norm:

\[
\max_{a \leq t \leq b} |f(t) - h(t)| \leq \max_{a \leq t \leq b} |f(t) - g(t)| + \max_{a \leq t \leq b} |g(t) - h(t)|.
\]

This, applied to the derivatives, gives also

\[
\max_{a \leq t \leq b} |f'(t) - h'(t)| \leq \max_{a \leq t \leq b} |f'(t) - g'(t)| + \max_{a \leq t \leq b} |g'(t) - h'(t)|.
\]

Adding the two inequalities, we get

\[
d(f, h) \leq d(f, g) + d(g, h).
\]

(b) For this part of the problem, let’s denote

\[
\rho(f, g) = \max_{a \leq t \leq b} |f(t) - g(t)|, \quad \text{for } f, g \in C[a, b].
\]

Then, for every two functions \( f, g \in C^1[a, b] \), we have

\[
\rho(D(f), D(g)) = \max_{a \leq t \leq b} |f'(t) - g'(t)|
\]

\[
\leq \max_{a \leq t \leq b} |f(t) - g(t)| + \max_{a \leq t \leq b} |f'(t) - g'(t)| = d(f, g).
\]

This shows that \( D : C^1[a, b] \to C[a, b] \) is Lipschitz continuous (and therefore, continuous).
Problem 3. Let $C[0,1]$ consist of the continuous functions on $[0,1]$ and furnished with the metric

$$d(f, g) = \int_0^1 |f(t) - g(t)| \, dt.$$ 

Define $T : C[0,1] \to \mathbb{R}$ by

$$T(f) = \int_0^1 f(t) \, dt.$$ 

Is $T$ continuous?

**Solution.** Yes, $T$ is continuous, and even Lipschitz continuous with a Lipschitz constant 1. For every $f, g \in C[0,1],

$$|T(f) - T(g)| = \left| \int_0^1 f(t) \, dt - \int_0^1 g(t) \, dt \right| = \left| \int_0^1 (f(t) - g(t)) \, dt \right| \leq \int_0^1 |f(t) - g(t)| \, dt = d(f, g).$$

Problem 4. Let a function $f : \mathbb{R} \to \mathbb{R}$ be defined by setting $f(1/n) = c_n$ for $n = 1, 2, 3, \ldots$ where $c_1, c_2, \ldots$ is a given sequence, and elsewhere $f(x) = 0$. Find a condition on that sequence so that $f'(0)$ exists.

**Solution.** By the definition of derivative, $f'(0)$ is defined as the following limit:

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x},$$

which must exist.

By the definition of $f$,

$$\frac{f(x)}{x} = \begin{cases} 
nc_n, & \text{if } x = 1/n \text{ for some } n = 1, 2, \ldots \\
0, & \text{otherwise.}
\end{cases}$$
In every neighborhood of 0 there are infinitely many \( x \)'s for which \( f(x) = 0 \). This means that, if \( \lim_{x \to 0} \frac{f(x)}{x} \) exists, it must be 0. On the other hand, if this limit exist, we must also have

\[
\lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{f(1/n)}{1/n} = \lim_{n \to \infty} nc_n.
\]

We see that \( f'(0) \) exists if and only if

\[
\lim_{n \to \infty} nc_n = 0,
\]

and if this is the case, \( f'(0) = 0 \).

The necessity of this condition was shown above. The fact that it is also sufficient, can be seen from the following:

Let an \( \varepsilon > 0 \) be given, and let \( N \) be such that for all \( n \geq N \) we have \( |nc_n| < \varepsilon \). Choose \( \delta = 1/N \). Then, whenever \( 0 < |x| < \delta \), we have either \( f(x)/x = 0 \) or \( f(x)/x = nc_n \) for some \( n > N \). In either case,

\[
\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| = \left| \frac{f(x)}{x} \right| < \varepsilon.
\]

\[\Box\]

**Problem 5.** Suppose \( f \) satisfies the hypotheses of the mean value theorem on \([a, b]\). Let \( S \) be the set of all slopes of chords determined by pairs of points on the graph of \( f \) and let

\[
D = \{ f'(x) : x \in (a, b) \}.
\]

(a) Prove that \( S \subseteq D \).

(b) Give an example to show that \( D \) can contain numbers not in \( S \).

**Solution.**

(a) Let \( s \in S \). Then there exist \( x, y \in [a, b] \) so that \( x < y \) and

\[
s = \frac{f(x) - f(y)}{x - y}.
\]
By the mean value theorem, there exists a \( \xi \in (x, y) \subset [a, b] \) such that

\[
f'(\xi) = \frac{f(x) - f(y)}{x - y}.
\]

Thus,

\[
s = f'(\xi) \in D.
\]

Since \( s \in S \) was arbitrary, we see that \( S \subset D \).

(b) Let \( [a, b] = [-1, 1] \), and take \( f(x) = x^3 \). Then, whenever \( x, y \in [a, b] \) and \( x < y \), we have \( f(x) < f(y) \), so

\[
\frac{f(x) - f(y)}{x - y} > 0,
\]

in particular, \( 0 \notin S \). On the other hand, \( 0 = f'(0) \in D \).

\[\square\]

**Problem 6.** Suppose \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\). If

\[
\lim_{x \to a^+} f'(x) = C
\]

what can you conclude about the right-hand derivative of \( f \) at \( a \)?

**Solution.** We can conclude that \( f'(a^+) \) exists and is equal to \( C \). We actually did this in class, in order to show that a derivative cannot have jump discontinuities. Here is how the argument went:

Let an \( \varepsilon > 0 \) be given, and let \( \delta > 0 \) be such that, whenever \( a < x < a + \delta \), we have

\[
|f'(x) - C| < \varepsilon.
\]

Then, for the very same \( \delta \), whenever \( a < x < a + \delta \), we can find a \( \xi \in (a, x) \) by the mean value theorem, so that

\[
f'(\xi) = \frac{f(x) - f(y)}{x - y},
\]

and therefore

\[
\left| \frac{f(x) - f(a)}{x - a} - C \right| = |f'(\xi) - C| < \varepsilon,
\]

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since \( a < \xi < x < a + \delta \). This proves that

\[
\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = C.
\]

**Problem 7.** Let \( f : [0, 1] \to \mathbb{R} \) be a differentiable function such that \( |f'(x)| \leq M \) for all \( x \in (0, 1) \). Show that

\[
\left| \int_0^1 f(x) \, dx - \frac{1}{n} \sum_{k=1}^n f \left( \frac{k}{n} \right) \right| \leq \frac{M}{n}.
\]

**Solution.** By the mean value theorem, for every \( x, y \in [0, 1] \), there exists a \( \xi \) such that

\[
f(x) - f(y) = f'({\xi}) (x - y),
\]

and hence

\[
|f(x) - f(y)| = |f'({\xi})| \cdot |x - y| \leq M |x - y|.
\]

We have:

\[
\left| \int_0^1 f(x) \, dx - \frac{1}{n} \sum_{k=1}^n f \left( \frac{k}{n} \right) \right| = \left| \sum_{k=1}^n \int_{k/(n-1)}^{(k+1)/n} \left( f(x) - f \left( \frac{k}{n} \right) \right) \, dx \right|
\]

\[
\leq \sum_{k=1}^n \int_{k/(n-1)}^{(k+1)/n} \left| f(x) - f \left( \frac{k}{n} \right) \right| \, dx \leq \sum_{k=1}^n \int_{k/(n-1)}^{(k+1)/n} M \left( \frac{k}{n} - x \right) \, dx
\]

\[
= M \sum_{k=1}^n \int_{k/(n-1)}^{(k+1)/n} \left( \frac{k}{n} - x \right) \, dx = M \sum_{k=1}^n \frac{1}{2n^2} = \frac{M}{2n}.
\]