Dept. of Math. Sci., WPI
MA 3832 Advanced Calculus - 2
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Homework Assignment 4

Solutions

Problem 1. Let $f_n \to f$ pointwise at every point in the interval $[a, b]$. We have seen that even if each f_n is continuous it does not follow that f is continuous. Which of the following statements are true?

(a) If each f_n is increasing on $[a, b]$, then so is f.

(b) If each f_n is nondecreasing on $[a, b]$, then so is f.

(c) If each f_n is bounded on $[a, b]$, then so is f.

(d) If each f_n is everywhere discontinuous on $[a, b]$, then so is f.

(e) If each f_n is constant on $[a, b]$, then so is f.

(f) If each f_n is positive on $[a, b]$, then so is f.

(g) If each f_n is linear on $[a, b]$, then so is f.

(h) If each f_n is convex on $[a, b]$, then so is f.

SOLUTION.

(a) Not true in general, because strict inequalities are not preserved by limits. Example: let $f_n(x) = x/n$. Then each f_n is strictly increasing, but f is not (because $f(x) = 0$ for all x).

(b) True. Let $x, y \in [a, b]$ be given, so that $x \leq y$. Then, for each n,

$$f_n(x) \leq f_n(y).$$

Letting $n \to \infty$, we get

$$f(x) \leq f(y).$$

(c) Not true in general. Example: let $[a, b] = [0, 1]$ and let

$$f_n(x) = \begin{cases} 1/x, & \text{if } x > 1/n \\ 0, & \text{otherwise.} \end{cases}$$

Then, for each n,

$$0 \leq f_n(x) \leq n,$$

i.e. each f_n is bounded. On the other hand,

$$f(x) = \begin{cases} 1/x, & \text{if } x > 0 \\ 0, & \text{if } x = 0, \end{cases}$$

and is therefore unbounded.

(d) Not true in general. Example: let

$$f_n(x) = \begin{cases} 1/n, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Then each f_n is everywhere discontinous, but the limit is the zero function, which is continuous on $[a, b]$

(e) True. Let an arbitrary $x \in [a, b]$ be fixed. For every n,

$$f_n(x) = f_n(a).$$

Letting $n \to \infty$, we get

$$f(x) = f(a).$$

Since x was arbitrary, we see that f is constant.

(f) Not true in general, because strict inequalities are not preserved by limits. Example: let f_n be the constant $1/n$.

(g) True. Indeed, if each f_n is linear, then there exist constants $m_1, m_2, \ldots,$ $m_n, \ldots$ and $c_1, c_2, \ldots, c_n, \ldots$ so that, for every n,

$$f_n(x) = m_n x + c_n \qquad \text{for all } x \in [a, b].$$

From the pointvise convergence,

$$f_n(a) \to f(a), \qquad f_n(b) \to f(b),$$

and therefore there exist limits

$$\begin{aligned}\lim_{n\to\infty} m_n &= \lim_{n\to\infty} \frac{f_n(b)-f_n(a)}{b-a} = \lim_{n\to\infty} \frac{f(b)-f(a)}{b-a} =: m,\\ \lim_{n\to\infty} c_n &= \lim_{n\to\infty} \frac{bf_n(a)-af_n(b)}{b-a} = \lim_{n\to\infty} \frac{bf(a)-af(b)}{b-a} =: c\end{aligned}$$

Then, for each $x \in [a,b]$,

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} m_n x + c_n = mx + c.$$

(f) True. Let $x, y \in [a,b]$ and $\alpha \in (0,1)$ be given. Then, for every n we have

$$f_n(\alpha x + (1-\alpha)x) \leq \alpha f_n(x) + (1-\alpha) f_n(y).$$

Letting $n \to \infty$, we get

$$f(\alpha x + (1-\alpha)x) \leq \alpha f(x) + (1-\alpha) f(y).$$

Problem 2. Prove that if $f_n \to f$ pointwise on a finite set D, then the convergence is uniform.
SOLUTION. Let $D = \{x_1, x_2, \ldots, x_m\}$. We know that

$$\lim_{n\to\infty} f_n(x_k) = f(x_k) \qquad \text{for } k = 1, \ldots, m.$$

Let an $\varepsilon > 0$ be given. Then, for each $k = 1, \ldots, m$ we can find an N_k so that, whenever $n \geq N_k$, we have

$$|f_n(x_k) - f(x_k)| < \varepsilon.$$

Choose $N = \max\{N_1, N_2, \ldots, N_m\}$. Then, whenever $n \geq N$, we have

$$|f_n(x_k) - f(x_k)| < \varepsilon \qquad \text{for all } k = 1, \ldots, m.$$

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Problem 3. Prove that if $f_n \to f$ uniformly on a set E, then $f_n \to f$ uniformly on every subset of E.

Solution. By definition, $f_n \to f$ uniformly on the set E, means that for every $\varepsilon > 0$, we can find an N so that, whenever $n \geq N$, the inequality

$$|f_n(x) - f(x)| < \varepsilon$$

holds for all $x \in E$. Of course it will hold for all x in any subset of E as well. ∎

Problem 4. Prove or disprove that if f is a continuous function on $(-\infty, \infty)$, then

$$f(x + 1/n) \to f(x)$$

uniformly on $(-\infty, \infty)$. What extra condition, stronger than continuity, would work if not?

Solution. The statement is not true in general, for example, if we take $f(x) = x^2$, we have

$$\sup_{x \in \mathbb{R}} |f(x + 1/n) - f(x)| = \sup_{x \in \mathbb{R}} (2x/n + 1/n^2) = \infty$$

for all n, which shows that there is no uniform convergence.

The following statement is correct: if f is *uniformly* continuous on $(-\infty, \infty)$, then

$$f(x + 1/n) \to f(x)$$

uniformly on $(-\infty, \infty)$.

Indeed, let an $\varepsilon > 0$ be given. Choose a $\delta > 0$ so that, whenever $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$. Choose $N > 1/\delta$. Then, whenever $n \geq N$, we have $|(x + 1/n) - x| < \delta$ and

$$|f(x + 1/n) - f(x)| < \varepsilon \qquad \text{for all } x \in \mathbb{R}.$$

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