Department of Mathematical Sciences
WPI
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Term D01 S O L U T I O N S

MA3832 MIDTERM EXAM
50 minutes

Problem	**Points**	**Score**
1	**15**	
2	**15**	
3	**15**	
4	**15**	
5	**15**	
6	**15**	
7	**10**	
Total	**100**	

- This is a 50-minute exam. It has seven (7) problems. It is your responsibity to make sure you have all the pages.
- Write your solutions in the space provided below each problem. If you need more space, use the back of the sheet (please indicate clearly if you do so).
- No books or notes are to be consulted.
- Calculators are NOT allowed.
- The solutions you submit must be your own work. You may not look at or copy the work of others.
- Show all your work. No credit will be given for unsupported answers.

GOOD LUCK

1. (15 pts.) On the set of real numbers, $\mathbb{R}$, consider the metric d given by the formula

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1 + |x - y|, & \text{if } x \neq y \end{cases}$$

(a) Verify that d is indeed a metric, by proving the triangle inequality for it (the other properties of metric are obvious).

(b) Describe which sets are both open and closed in this metric.

Solution.

(a) Let arbitrary $x, y, z \in \mathbb{R}$ be given. We need to show that

$$d(x,z) \leq d(x,y) + d(y,z).$$

When $x = z$, the left hand side is 0, and the inequality is obvious. Let's consider now the case when $x \neq z$. Then, we cannot have both $y = x$ and $y = z$. Since x and z appear in the formula in a symmetric way, let's assume, without loss of generality, that $y \neq x$. Then

$$\begin{aligned} d(x,z) &= 1 + |x - z| \\ d(x,y) &= 1 + |x - y| \\ d(y,z) &\geq |y - z| \end{aligned}$$

The last inequality is correct, regardless of whether $y = z$ or $y \neq z$. Adding the last two inequalities, we get

$$d(x,y) + d(y,z) \geq 1 + |x - y| + |y - z| \geq 1 + |x - z| = d(x,z).$$

(b) First of all, observe that, for each point x, the open ball of radius 1 centered at this point, contains no other points, i.e.

$$B(x,1) = \{x\}, \qquad \text{for all } x \in \mathbb{R}.$$

This shows, in particular, that each point is an open set. Since every set can be represented as the union of the points that constitute it, we see that every set is open. Finally, the complement of each set (being again a set in this metric space) must also be open. Thus, we see that *every* subset $A \subset \mathbb{R}$ is both open and closed under the metric d.

2. (15 pts.) Let X be a metric space with a metric d, such that every subset $A \subset X$ is both open and closed. Consider also $\mathbb{R}$ with the usual distance, and let $f : \mathbb{R} \to X$ be a continuous mapping. Show that f must be a constant.
Solution. Denote $x_0 := f(0) \in X$. Then the set $f^{-1}(\{x_0\})$ is a subset of $\mathbb{R}$ which is open, closed and nonempty. Thus

$$f^{-1}(\{x_0\}) = \mathbb{R},$$

or, in other fords, $f(t) = f(0)$ for all $t \in \mathbb{R}$.

3. (15 pts.) Let X be a metric space with a metric d, and let $A \subset X$ be dense in X. Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be two continuous functions such that $f(x) = g(x)$ for all $x \in A$. Show that $f(x) = g(x)$ for all $x \in X$.
Solution.

- Here is one possible solution. Let an arbitrary $x \in X$ be given. Since A is dense in X, there exists a sequence $x_1, x_2, \ldots, x_n, \ldots$ such that $x_n \to x$ and $x_n \in A$ for all n. Then, since f and g are continuous on X,

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(x).$$

- Here is another possible solution. Consider the function $h = f - g$. This is a continuous function on X, such that $h(A) = \{0\}$. Therefore, $h^{-1}(\{0\})$ is a closed subset of X which includes A. Hence $h^{-1}(\{0\}) \supset \overline{A}$. Since A is dense in X, $\overline{A} = X$ and hence $h^{-1}(\{0\}) = X$. This means that h is identically 0 on all X.

4. (15 pts.) Show that the equation

$$x^4 - 2x - 11 = 0$$

has at least one real root. (Do not attempt to find the root)

Solution. Denote $f(x) = x^4 - 2x - 11$ for $x \in \mathbb{R}$. Then f, being a polynomial, is continuous on $\mathbb{R}$. Observe that

$$\begin{aligned} f(0) &= -11 < 0 \\ f(2) &= 16 - 4 - 11 = 1 > 0 \end{aligned}$$

By the Intermediate Value Property of continuous functions, there exists a number $c \in [0, 2]$ such that $f(c) = 0$.

5. (15 pts.) Let f be given by the formula

$$f(x) = \frac{1}{x}, \qquad x \geq 1.$$

Show that f is uniformly continuous on $[1, \infty)$.
Soultion. Let an $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$. Then, whenever $x, y \in [1, \infty)$ and $|x - y| < \delta$, we have

$$|f(x) - f(y)| = |\frac{1}{x} - \frac{1}{y}| = \frac{|x-y|}{xy} \leq |x - y| < \delta < \varepsilon.$$

6. (15 pts.) Let f be given by the formula

$$f(x) = x^2, \qquad x \geq 1.$$

Show that f is **not** uniformly continuous on $[1, \infty)$.
Solution. Choose $\varepsilon := 1$. For every given $\delta > 0$, denote

$$x = \frac{1}{\delta}, \qquad , y = \frac{1}{\delta} + \frac{\delta}{2}.$$

We have

$$|x - y| = \frac{\delta}{2} < \delta,$$

yet

$$|f(x) - f(y)| > 1 = \varepsilon.$$

7. (10 pts.) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that the composition $f \circ f$ is continuous. Does it follow that f is continuous as well? (Justify your answer)
Solution. No. Consider the following example:

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational,} \\ -x, & \text{if } x \text{ is irrational,} \end{cases}$$

Clearly, this is not a continuous function, yet $f(f(x)) = x$ for all x, so $f \circ f$ is continuous.