Dept. of Math. Sci., WPI
MA 571 Financial Mathematics I
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Homework Assignment 4

Solutions

Problem 1. Give an example of a non-empty set Ω and a σ-algebra $\mathcal{F}$ of subsets of Ω such that $\mathcal{F}$ has exactly three elements, or prove that such a pair $(\Omega, \mathcal{F})$ does not exist.

Solution. Such a pair does not exist.

Indeed, suppose that Ω is a non-empty set and $\mathcal{F}$ is a σ-algebra on Ω with exactly three elements. From the definition of a σ-algebra, two of these elements *must* be $\emptyset$ and Ω. Further, there must be exactly one more element of $\mathcal{F}$, let's call it A, such that $A \neq \Omega$ and $A \neq \emptyset$. Since $\mathcal{F}$ is a σ-algebra, A^c must be an element of $\mathcal{F}$, too. But since

$$\mathcal{F} = \{\emptyset, \Omega, A\},$$

one of the following must be true:

$$\begin{aligned} A^c &= \emptyset, \\ A^c &= \Omega, \\ A^c &= A. \end{aligned}$$

However, none of the three is possible.

Problem 2. Let Ω_3 be the sample space of the model of a coin being tossed three times,

$$\Omega_3 = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Suppose you are told the number of heads obtained in the three tosses. We will say that a certain event (subset of Ω_3) is resolved by this piece of information, if the information given is sufficient to tell whether or not the event has occured. Thus, the set $\{HHH\}$ is resolved by being told the number of heads in the three tossings, but the event $\{HTT\}$ is not.

(i) Make a list of all the sets resolved by this information. (There are four "fundamental" ones, and sixteen altogether.)

(ii) Is the collection of sets you listed in (i) a σ-algebra?

Solution.

(i) For $k = 0, 1, 2, 3$, let A_k denote the event that there are exactly k heads in the three tosses, i.e.,

$$\begin{aligned} A_0 &:= \{TTT\}, \\ A_1 &:= \{TTH, THT, HTT\}, \\ A_2 &:= \{HHT, HTH, THH\}, \\ A_3 &:= \{HHH\}. \end{aligned}$$

Observe that the sets A_0, A_1, A_2 and A_3 are disjoint. These are the four "fundamental" sets. A complete list of the sixteen sets resolved by knowing the number of heads is:

$$\begin{array}{ll} \emptyset, & \Omega \\ A_0, & A_1 \cup A_2 \cup A_3 \\ A_1, & A_0 \cup A_2 \cup A_3 \\ A_2, & A_0 \cup A_1 \cup A_3 \\ A_3, & A_0 \cup A_1 \cup A_2 \\ A_0 \cup A_1, & A_2 \cup A_3 \\ A_0 \cup A_2, & A_1 \cup A_3 \\ A_0 \cup A_3, & A_1 \cup A_2. \end{array}$$

Denote the family of these 16 sets by $\mathcal{G}$.

(ii) Yes, $\mathcal{G}$ is a σ-algebra. We check the three conditions:

(a) $\emptyset \in \mathcal{G}$.

(b) Complements: look at the list above. It is organized in such a way that on each line the two sets are complements to each other.

(c) Countable unions: each of the sets in $\mathcal{G}$ is a union of some of the sets A_0, A_1, A_2 and A_3. Their unions will be again of this form. All sets of this form are in $\mathcal{G}$.

Problem 3. John and Peter take turns tossing a fair coin. The first one to get a tail keeps the coin.

(i) If John goes first, what are his chances to win?

(ii) What is the probability that it will take no more than four tossings for the game to end?

Solution. Denote, for each integer $k \geq 1$,

$$\begin{aligned} A_k &:= \{\text{for the first time, a tail appears at the } k\text{-th tossing}\} \\ &= \{\omega_1 = \cdots = \omega_{k-1} = H, \omega_k = T\}. \end{aligned}$$

Then, for each $k \geq 1$,

$$\mathbf{P}(A_k) = \frac{1}{2^k}.$$

Observe that the sets A_k are disjoint.

(i) The probability of John winning is

$$\mathbf{P}\left(\bigcup_{k\text{-odd}} A_k\right) = \sum_{k\text{-odd}} \mathbf{P}(A_k) = \sum_{m=0}^{\infty} \mathbf{P}(A_{2m+1}) = \sum_{m=0}^{\infty} \frac{1}{2^{2m+1}} = \frac{2}{3}.$$

(ii) The probability that the game will last at most four tossings is

$$\begin{aligned} \mathbf{P}(\bigcup_{k=1}^{4} A_k) &= \sum_{k=1}^{4} \mathbf{P}(A_k) \\ &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}. \end{aligned}$$

Problem 4. Let $\mathcal{G}$ be a σ-algebra of subsets of a nonempty set Ω. Show the following properties:

(i) If $A_1, A_2, \ldots, A_n$ is a finite sequence of sets in $\mathcal{G}$, then the union $\cup_{k=1}^{n} A_k$ and the intersection $\cap_{k=1}^{n} A_k$ are also in $\mathcal{G}$.

(ii) If A and B are sets in $\mathcal{G}$, then their set-theoretic difference $A \setminus B := A \cap B^c$ is also in $\mathcal{G}$.

(iii) If A and B are sets in $\mathcal{G}$, then their symmetric difference $A \triangle B := (A \setminus B) \cup (B \setminus A)$ is also in $\mathcal{G}$.

Solution. It is given that $\mathcal{G}$ is a σ-algebra. By definition this means that

(a) $\emptyset \in \mathcal{G}$;

(b) if $A \in \mathcal{G}$, then $A^c \in \mathcal{G}$;

(c) if $A_1, A_2, \ldots, A_n, \ldots \in \mathcal{G}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{G}$.

We now prove the statements of the problem:

(i) Let $A_1, A_2, \ldots, A_n \in \mathcal{G}$ be given. If we choose $A_{n+1} = A_{n+2} = \cdots = \emptyset$, then,

$$\bigcup_{k=1}^{n} A_k = \bigcup_{k=1}^{\infty} A_k \in \mathcal{G},$$

by (c). On the other hand, if we choose $A_{n+1} = A_{n+2} = \cdots = \Omega$, then,

$$\bigcap_{k=1}^{n} A_k = \bigcap_{k=1}^{\infty} A_k \in \mathcal{G}.$$

(ii) If $A, B \in \mathcal{G}$, then, by (b), $B^c \in \mathcal{G}$, and, by (i), $A \cap B^c \in \mathcal{G}$.

(iii) If $A, B \in \mathcal{G}$, then, by (ii), $A \setminus B, B \setminus A \in \mathcal{G}$, and, by (i), $(A \setminus B) \cup (B \setminus A) \in \mathcal{G}$.

Problem 5. In this problem, we look again at the probability space $[0, 1]$ with Lebesgue measure. Consider the following generalization of the Cantor set. Let $p_1, p_2, \ldots, p_n, \ldots$ be real numbers, $0 < p_n < 1$ for each n. Start with $C_0 = [0, 1]$. Remove an open interval of length p_1 in the middle, so that you are left with two equal closed intervals of length $(1 - p_1)/2$ each. Call the resulting set C_1. From the middle of each of the two components of C_1, remove an open interval, whose length is p_2 times the length of the component, i.e. $p_2(1 - p_1)/2$. Call the result C_2. The set C_2 consists of four disjoint closed intervals, each of length $(1 - p_1)(1 - p_2)/4$. Continue this process indefinitely. At stage k, we have a set C_k, consisting of 2^k pieces (closed intervals). Define $C := \bigcap_{k=1}^{\infty} C_k$. The set C is topologically equivalent to the Cantor set described in lectures. The usual ternary Cantor set is obtained if $p_n = 1/3$ for all n.

(i) For each k, show that

$$\mathbf{P}(C_k) = (1 - p_1)(1 - p_2) \cdots (1 - p_k).$$

(ii) Let $p_n = 1/(n + 1)^2$. What is $\mathbf{P}(C)$ in this case?

(iii) Let a real number α be given, with $0 \leq \alpha < 1$. Construct a sequence of numbers $p_1, p_2, \ldots, p_n, \ldots$, with $0 < p_n < 1$ for each n, in such a way that $\mathbf{P}(C) = \alpha$.

Soluition.

(i) For $k = 1$, observe that C_1 is obtained from the interval $[0, 1]$ by removing an interval of length p_1. Hence,

$$\mathbf{P}(C_1) = (1 - p_1).$$

Similarly, C_{k+1} is obtained from C_k by removing, from each of its parts, an interval whose length is p_{k+1} times the length of that part. Thus,

$$\mathbf{P}(C_{k+1}) = (1 - p_{k+1})\mathbf{P}(C_k).$$

The proof of (i) is then finished by induction.

(ii)

$$\begin{aligned}
\mathbf{P}(C) &= \lim_{k\to\infty} \mathbf{P}(C_k) \\
&= \lim_{k\to\infty} (1 - \frac{1}{2^2})(1 - \frac{1}{3^2}) \cdots (1 - \frac{1}{(k+1)^2}) \\
&= \lim_{k\to\infty} \frac{(1)(3)}{(2)(2)} \frac{(2)(4)}{(3)(3)} \cdots \frac{(k)(k+2)}{(k+1)(k+1)} \\
&= \lim_{k\to\infty} \frac{1}{2} \cdot \frac{(k+2)}{(k+1)} \\
&= \frac{1}{2}.
\end{aligned}$$

(iii) If $\alpha = 0$, we can take $p_n = 1/3$ for all n. We showed in class that in this case $\mathbf{P}(C) = 0$.

If $0 < \alpha < 1$, we can choose, for example,

$$p_n := 1 - \sqrt[2^n]{\alpha} = 1 - \alpha^{(1/2^n)}.$$

There are other possibilties, too, actually infinitely many.

Problem 6. Let $(\Omega, \mathcal{F})$ be a measurable space and let μ be a function that maps $\mathcal{F}$ into $[0, \infty)$ with the following properties:

(a) If A_1 and A_2 are disjoint sets in $\mathcal{F}$, then $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$.

(b) If $A_1, A_2, \ldots$ is a sequence of sets in $\mathcal{F}$, then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mu(A_k).$$

Show that μ is a σ-additive measure, i.e., show that if $A_1, A_2, \ldots$ is a sequence of disjoint sets in $\mathcal{F}$, then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k).$$

Solution. Let $A_1, A_2, \ldots$ be a sequence of disjoint sets in $\mathcal{F}$. For every n we have:

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \geq \mu\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} \mu(A_k).$$

Since the left-hand side does not depend on n and the inequality holds for all n, we can take supremums of both sides and conclude that

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \geq \sum_{k=1}^{\infty} \mu(A_k).$$

Combining this with (b) gives us the desired equality.

Problem 7. Let $\Omega = \mathbb{N}$, the natural numbers. Let $\mathcal{F}$ be the collection of all subsets of Ω. For every set $A \in \mathcal{F}$, let $\mathbf{1}_A : \Omega \to \{0, 1\}$ be its indicator function, i.e.

$$\mathbf{1}_A(i) = \begin{cases} 0, & \text{if } i \notin A \\ 1, & \text{if } i \in A. \end{cases}$$

Let $\mu : \mathcal{F} \to [0, \infty)$ be given by

$$\mu(A) := \sum_{i=1}^{\infty} \frac{1}{2^i} \mathbf{1}_A(i).$$

Is μ a probability measure on $(\Omega, \mathcal{F})$?

Solution. The answer is yes.

By definition, μ is a probability measure on $(\Omega, \mathcal{F})$ if it satisfies the following conditions:

(i) μ is defined on $\mathcal{F}$ and takes only non-negative values,

(ii) $\mu(\Omega) = 1$,

(iii) μ is countably-additive.

We verify these conditions one by one.

The condition (i) is automatic from the definition of μ. Condition (ii) is also easy. Putting $A = \Omega = I\!N$ in the formula for μ we observe that $\mathbf{1}_\Omega(i) = 1$ for all $i \in I\!N$ and so

$$\mu(\Omega) := \sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1/2}{1 - 1/2} = 1,$$

by the formula for the geometric series.

Thus, the only non-trivial thing to verify is the countable additivity of μ. Let A_n, $n = 1, 2, ...$ be a countable sequence of pairwise disjoint subsets of Ω, i.e. $A_m \cap A_n = \emptyset$ whenever $m \neq n$. Let $A := \bigcup_{n=1}^{\infty} A_n$ and observe that

$$\mathbf{1}_A = \sum_{n=1}^{\infty} \mathbf{1}_{A_n}.$$

Further,

$$\mu(A) = \sum_{i=1}^{\infty} \frac{1}{2^i} \mathbf{1}_A(i) = \sum_{i=1}^{\infty} \frac{1}{2^i} \sum_{n=1}^{\infty} \mathbf{1}_{A_n}(i) = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{2^i} \mathbf{1}_{A_n}(i),$$

while

$$\sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \frac{1}{2^i} \mathbf{1}_{A_n}(i).$$

Since all the terms of the double series are non-negative, we can interchange the order of summation and arrive at

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A_n)$$

as needed, i.e. we proved the countable additivity of μ.

Remark. For double series whose terms change sign, the equality

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$$

is not always true. Here is an example. Let

$$a_{ij} = \begin{cases} 1, & \text{if } i = j, \\ -1, & \text{if } j = i+1, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = 0 \neq 1 = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$