Dept. of Math. Sci., WPI
MA 571 Financial Mathematics I
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Homework Assignment 5

Solutions

Problem 1. Let $\Omega = I\!\!R$ and let $X : \Omega \to I\!\!R$ be given by

$$X(\omega) = \begin{cases} \omega & \text{if } \omega \geq 0, \\ 0 & \text{if } \omega < 0. \end{cases}$$

(a) Find a σ-algebra, Σ_1 on Ω (other than the Borel σ-algebra or the power set) such that X is Σ_1-measurable.
(b) Find a σ-algebra, Σ_2 on Ω (other than the trivial algebra $\{I\!\!R, \emptyset\}$) such that X is not Σ_2-measurable.
Solution.
(a) We know that if we take $\Sigma_1 := \sigma(X)$, this will make X automatically Σ_1-measurable. This is (by definition) the smallest σ-algebra that will do the job. Thus, it is enough to make sure that Σ_1 is not the Borel σ-algebra on $I\!\!R$ and that it is not the power set of $I\!\!R$. To this end, let's just figure out what Σ_1 exactly is. For a single random variable X we have

$$\Sigma_1 = \sigma(\{X\}) = \{X^{-1}(B) : B \in \mathcal{B}(I\!\!R)\}.$$

For a Borel set $B \in \mathcal{B}(I\!\!R)$ we have, because of the particular form of X,

$$X^{-1}(B) = \begin{cases} B \cap (0, \infty) & \text{if } 0 \notin B, \\ B \cup (-\infty, 0] & \text{if } 0 \in B. \end{cases}$$

Thus,

$$\Sigma_1 = \sigma(\{X\}) = \{B \cap (0, \infty)\ ,\ \ B \cup (-\infty, 0]\ \ : B \in \mathcal{B}(I\!\!R)\}.$$

(b) Many answers are possible, here is just one possibility:

$$\Sigma_2 := \{\emptyset, I\!\!R, (-\infty, 0], (0, \infty)\}.$$

Since $(0, 1) \in \mathcal{B}(I\!\!R)$ and

$$X^{-1}(\ (0, 1)\ ) = (0, 1) \notin \Sigma_2,$$

it is clear that X is not Σ_2-measurable.

Problem 2. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let $X : \Omega \to I\!R$ be a random variable. Consider the function $F : I\!R \longrightarrow [0,1]$ defined by

$$F(x) := \mathbf{P}\{X \leq x\} = \mathbf{P}\{\omega \in \Omega : X(\omega) \leq x\}.$$

(F is called the c.d.f. of X.)

(i) Show that F is monotonic, and hence, Borel-measurable.

(ii) Let $Y : \Omega \longrightarrow I\!R$ be defined by

$$Y := F(X).$$

Explain why Y is a random variable.

(iii) Assuming that $F : I\!R \longrightarrow I\!R$ is continuous, compute $\mathbf{P}\{Y \leq \frac{1}{2}\}$.

Solution.

(i) Let $x' < x''$. Then, $(-\infty, x'] \subset (-\infty, x'']$ and therefore,

$$\mathbf{P}\{X \in (-\infty, x']\} \leq \mathbf{P}\{X \in (-\infty, x'']\},$$

which means exactly that $F(x') \leq F(x'')$, i.e. F is monotonically non-decreasing. All monotonic functions are Borel-measurable.

(ii) Since F is Borel-measurable and X is a random variable, $Y = F(X)$ is also a random variable.

(iii) Now assume that F is a continuous function. Define

$$\kappa := \sup\left\{x : F(x) \leq \frac{1}{2}\right\}.$$

Then, from the continuity of F it follows that $F(\kappa) = \frac{1}{2}$.

Further, observe that $x \leq \kappa$ implies that $F(x) \leq \frac{1}{2}$, because F is monotonic. Also, $x > \kappa$ implies that $F(x) > \frac{1}{2}$, because of the way we defined κ. Thus, $x \leq \kappa$ if and only if $F(x) \leq \frac{1}{2}$, so

$$\mathbf{P}\left\{Y \leq \frac{1}{2}\right\} = \mathbf{P}\left\{F(X) \leq \frac{1}{2}\right\} = \mathbf{P}\{X \leq \kappa\} = F(\kappa) = \frac{1}{2}.$$

Remark. Exactly in the same way it can be shown that, under the assumption of continuity of F, for every $y \in [0,1]$,

$$\mathbf{P}\{Y \le y\} = y,$$

i.e., the random variable Y is uniformly distributed on the interval $[0,1]$.

Problem 3. Toss a fair ($p = q = \frac{1}{2}$) coin repeatedly infinitely many times. Let ω_k denote the outcome on the k^{th} toss.

For each positive integer k, define

$$Y_k(\omega) = \begin{cases} 1 & \text{if } \omega_k = H, \\ 0 & \text{if } \omega_k = T, \end{cases}$$

and set

$$X(\omega) := 2\sum_{k=1}^{\infty} \frac{Y_k(\omega)}{3^k}.$$

Let $\mathcal{L}_X$ be the measure induced on $I\!R$ by X.

(i) Is there any point $a \in I\!R$ such that $\mathcal{L}_X\{a\} > 0$? Explain your answer.

(ii) Is there a subset C of $I\!R$ such that $\mathcal{L}_X(C) = 1$ but the Lebesgue measure of C is zero? Explain your answer.

Solution.

(i) For each $\omega \in \Omega$, we have $\mathbf{P}\{\omega\} = 0$. If $a \in I\!R$, then there can be at most one $\omega \in \Omega$ satisfying $X(\omega) = a$.

Indeed, suppose $X(\omega) = X(\omega')$ and $Y_1(\omega) < Y_1(\omega')$. Then $Y_1(\omega) = 0$, $Y_1(\omega') = 1$ and therefore $X(\omega) \le \frac{1}{3}$ while $X(\omega') \ge \frac{2}{3}$, which contradicts the fact that $X(\omega) = X(\omega')$. Thus, we see that, if $X(\omega) = X(\omega')$, then $Y_1(\omega) = Y_1(\omega')$. Proceeding further in a similar way by induction, we see that $Y_k(\omega) = Y_k(\omega')$ for all k. This implies $\omega = \omega'$. Then,

$$\mathcal{L}_X\{a\} = \mathbf{P}\{\omega : X(\omega) = a\} = 0 \text{ for every } a \in I\!R.$$

(ii) From the definition of X, it is clear that the range of X is a subset of $\mathbb{R}$ consisting of exactly those points α that can be represented in the form

$$\alpha = \sum_{k=1}^{\infty} \frac{\alpha_k}{3^k}$$

with $\alpha_k = 0$ or 2. From the discussion in class, we recall that this is exactly the ternary Cantor set C we constructed.

Thus,

$$\mathcal{L}_X(C) = \mathbf{P}\{\omega : X(\omega) \in C\} = 1.$$

while, as we showed in class, $\mu_0(C) = 0$.

We say that $\mathcal{L}_X$ and Lebesgue measure μ_0 are *singular* with respect to each other, because each puts all its measure on a set where the other puts no measure. Lebesgue measure puts all its measure on the complement of the Cantor set C; $\mathcal{L}_X$ puts all its measure on C.

Problem 4. Let μ_0 be the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and let the function $f : \mathbb{R} \longrightarrow [0, \infty]$ be a non-negative Borel function (we allow f to take infinite values for some values of x).

(a) If

$$C := \int_{\mathbb{R}} f \, d\mu_0 < \infty,$$

show that the set $A := \{x : f(x) = \infty\}$ has Lebesgue measure 0. [**Hint:** Consider the sets $A_n := \{x : f(x) > n\}$.]

(b) If

$$\int_{\mathbb{R}} f \, d\mu_0 = 0,$$

show that the set $B := \{x : f(x) > 0\}$ has Lebesgue measure 0. [**Hint:** Consider the sets $B_n := \{x : f(x) > \frac{1}{n}\}$.]

Solution. We proceed as follows.

(a) Define $A_n := \{x : f(x) > n\}$. Then

$$A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots \qquad \text{and} \qquad A = \bigcap_{n=1}^{\infty} A_n.$$

For every n,

$$\begin{aligned} C &= \int_{\mathbb{R}} f\, d\mu_0 = \int_{\mathbb{R}} f \cdot (\mathbf{1}_{A_n} + \mathbf{1}_{A_n^c}) d\mu_0 \\ &= \int_{\mathbb{R}} f \cdot \mathbf{1}_{A_n} d\mu_0 + \int_{\mathbb{R}} f \cdot \mathbf{1}_{A_n^c} d\mu_0 \geq \int_{\mathbb{R}} f \cdot \mathbf{1}_{A_n} d\mu_0 \\ &\geq \int_{\mathbb{R}} n \cdot \mathbf{1}_{A_n} d\mu_0 = n\, \mu_0(A_n), \end{aligned}$$

and hence

$$\mu_0(A_n) \leq \frac{C}{n}.$$

Finally,

$$\mu_0(A) = \mu_0(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu_0(A_n) \leq \lim_{n\to\infty} \frac{C}{n} = 0.$$

(b) Define $B_n := \{x : f(x) > \frac{1}{n}\}$. Then

$$B_1 \subseteq B_2 \subseteq \ldots \subseteq B_n \subseteq \ldots \qquad \text{and} \qquad B = \bigcup_{n=1}^{\infty} B_n.$$

For every n,

$$\begin{aligned} 0 &= \int_{\mathbb{R}} f\, d\mu_0 = \int_{\mathbb{R}} f \cdot (\mathbf{1}_{B_n} + \mathbf{1}_{B_n^c}) d\mu_0 \\ &= \int_{\mathbb{R}} f \cdot \mathbf{1}_{B_n} d\mu_0 + \int_{\mathbb{R}} f \cdot \mathbf{1}_{B_n^c} d\mu_0 \geq \int_{\mathbb{R}} f \cdot \mathbf{1}_{B_n} d\mu_0 \\ &\geq \int_{\mathbb{R}} \frac{1}{n} \cdot \mathbf{1}_{B_n} d\mu_0 = \frac{1}{n} \mu_0(B_n), \end{aligned}$$

and hence

$$\mu_0(B_n) = 0.$$

Finally,

$$\mu_0(B) = \mu_0(\bigcup_{n=1}^{\infty} B_n) = \lim_{n\to\infty} \mu_0(B_n) = 0.$$