Dept. of Math. Sci., WPI
MA 571 Financial Mathematics I
Instructor: Bogdan Doytchinov, Fall Term 2003

Homework Assignment 4

Due Thursday, October 2, 2003

Problem 1. Give an example of a non-empty set Ω and a σ-algebra $\mathcal{F}$ of subsets of Ω such that $\mathcal{F}$ has exactly three elements, or prove that such a pair $(\Omega, \mathcal{F})$ does not exist.

Problem 2. Let Ω_3 be the sample space of the model of a coin being tossed three times,

$$\Omega_3 = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Suppose you are told the number of heads obtained in the three tosses. We will say that a certain event (subset of Ω_3) is resolved by this piece of information, if the information given is sufficient to tell whether or not the event has occured. Thus, the set $\{HHH\}$ is resolved by being told the number of heads in the three tossings, but the event $\{HTT\}$ is not.

(i) Make a list of all the sets resolved by this information. (There are four "fundamental" ones, and sixteen altogether.)

(ii) Is the collection of sets you listed in (i) a σ-algebra?

Problem 3. John and Peter take turns tossing a fair coin. The first one to get a tail keeps the coin.

(i) If John goes first, what are his chances to win?

(ii) What is the probability that it will take no more than four tossings for the game to end?

Problem 4. Let $\mathcal{G}$ be a σ-algebra of subsets of a nonempty set Ω. Show the following properties:

(i) If $A_1, A_2, \dots, A_n$ is a finite sequence of sets in $\mathcal{G}$, then the union $\cup_{k=1}^n A_k$ and the intersection $\cap_{k=1}^n A_k$ are also in $\mathcal{G}$.

(ii) If A and B are sets in $\mathcal{G}$, then their set-theoretic difference $A \setminus B := A \cap B^c$ is also in $\mathcal{G}$.

(iii) If A and B are sets in $\mathcal{G}$, then their symmetric difference $A \triangle B := (A \setminus B) \cup (B \setminus A)$ is also in $\mathcal{G}$.

Problem 5. In this problem, we look again at the probability space $[0,1]$ with Lebesgue measure. Consider the following generalization of the Cantor set. Let $p_1, p_2, \dots, p_n, \dots$ be real numbers, $0 < p_n < 1$ for each n. Start with $C_0 = [0,1]$. Remove an open interval of length p_1 in the middle, so that you are left with two equal closed intervals of length $(1-p_1)/2$ each. Call the resulting set C_1. From the middle of each of the two components of C_1, remove an open interval, whose length is p_2 times the length of the component, i.e. $p_2(1-p_1)/2$. Call the result C_2. The set C_2 consists of four disjoint closed intervals, each of length $(1-p_1)(1-p_2)/4$. Continue this process indefinitely. At stage k, we have a set C_k, consisting of 2^k pieces (closed intervals). Define $C := \bigcap_{k=1}^\infty C_k$. The set C is topologically equivalent to the Cantor set described in lectures. The usual ternary Cantor set is obtained if $p_n = 1/3$ for all n.

(i) For each k, show that

$$\mathbf{P}(C_k) = (1-p_1)(1-p_2)\cdots(1-p_k).$$

(ii) Let $p_n = 1/(n+1)^2$. What is $\mathbf{P}(C)$ in this case?

(iii) Let a real number α be given, with $0 \le \alpha < 1$. Construct a sequence of numbers $p_1, p_2, \dots, p_n, \dots$, with $0 < p_n < 1$ for each n, in such a way that $\mathbf{P}(C) = \alpha$.

Problem 6. Let $(\Omega, \mathcal{F})$ be a measurable space and let μ be a function that maps $\mathcal{F}$ into $[0, \infty)$ with the following properties:

(a) If A_1 and A_2 are disjoint sets in $\mathcal{F}$, then $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$.

(b) If $A_1, A_2, \ldots$ is a sequence of sets in $\mathcal{F}$, then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mu(A_k).$$

Show that μ is a σ-additive measure, i.e., show that if $A_1, A_2, \ldots$ is a sequence of disjoint sets in $\mathcal{F}$, then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k).$$

Problem 7. Let $\Omega = \mathbb{N}$, the natural numbers. Let $\mathcal{F}$ be the collection of all subsets of Ω. For every set $A \in \mathcal{F}$, let $\mathbf{1}_A : \Omega \to \{0, 1\}$ be its indicator function, i.e.

$$\mathbf{1}_A(i) = \begin{cases} 0, & \text{if } i \notin A \\ 1, & \text{if } i \in A. \end{cases}$$

Let $\mu : \mathcal{F} \to [0, \infty)$ be given by

$$\mu(A) := \sum_{i=1}^{\infty} \frac{1}{2^i} \mathbf{1}_A(i).$$

Is μ a probability measure on $(\Omega, \mathcal{F})$?