Dept. of Math. Sci., WPI
MA 571 Financial Mathematics I
Instructor: Bogdan Doytchinov, Fall Term 2003

Homework Assignment 5

Due Thursday, October 9, 2003

Problem 1. Let $\Omega = I\!R$ and let $X : \Omega \to I\!R$ be given by

$$X(\omega) = \begin{cases} \omega \text{ if } \omega \geq 0, \\ 0 \text{ if } \omega < 0. \end{cases}$$

(a) Find a σ-algebra, Σ_1 on Ω (other than the Borel σ-algebra or the power set) such that X is Σ_1-measurable.
(b) Find a σ-algebra, Σ_2 on Ω (other than the trivial algebra $\{I\!R, \emptyset\}$) such that X is not Σ_2-measurable.

Problem 2. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and let $X : \Omega \to I\!R$ be a random variable. Consider the function $F : I\!R \longrightarrow [0, 1]$ defined by

$$F(x) := \mathbf{P}\{X \leq x\} = \mathbf{P}\{\omega \in \Omega : X(\omega) \leq x\}.$$

(F is called the c.d.f. of X.)

(i) Show that F is monotonic, and hence, Borel-measurable.

(ii) Let $Y : \Omega \longrightarrow I\!R$ be defined by

$$Y := F(X).$$

Explain why Y is a random variable.

(iii) Assuming that $F : I\!R \longrightarrow I\!R$ is continuous, compute $\mathbf{P}\{Y \leq \frac{1}{2}\}$.

Problem 3. Toss a fair ($p = q = \frac{1}{2}$) coin repeatedly infinitely many times. Let ω_k denote the outcome on the k^{th} toss.

For each positive integer k, define

$$Y_k(\omega) = \begin{cases} 1 & \text{if } \omega_k = H, \\ 0 & \text{if } \omega_k = T, \end{cases}$$

and set

$$X(\omega) := 2\sum_{k=1}^{\infty} \frac{Y_k(\omega)}{3^k}.$$

Let $\mathcal{L}_X$ be the measure induced on $I\!R$ by X.

(i) Is there any point $a \in I\!R$ such that $\mathcal{L}_X\{a\} > 0$? Explain your answer.

(ii) Is there a subset C of $I\!R$ such that $\mathcal{L}_X(C) = 1$ but the Lebesgue measure of C is zero? Explain your answer.

Problem 4. Let μ_0 be the Lebesgue measure on $(I\!R, \mathcal{B}(I\!R))$ and let the function $f : I\!R \longrightarrow [0, \infty]$ be a non-negative Borel function (we allow f to take infinite values for some values of x).

(a) If

$$C := \int_{I\!R} f \, d\mu_0 < \infty,$$

show that the set $A := \{x : f(x) = \infty\}$ has Lebesgue measure 0. [**Hint:** Consider the sets $A_n := \{x : f(x) > n\}$.]

(b) If

$$\int_{I\!R} f \, d\mu_0 = 0,$$

show that the set $B := \{x : f(x) > 0\}$ has Lebesgue measure 0. [**Hint:** Consider the sets $B_n := \{x : f(x) > \frac{1}{n}\}$.]