**Dept. of Math. Sci., WPI**
**MA 571 Financial Mathematics I**
**Instructor: Bogdan Doytchinov, Fall Term 2003**

# Homework Assignment 7

Due Thursday, November 6, 2003

**Problem 1.** (Martingale transform.) Toss a fair ($p = q = \frac{1}{2}$) coin repeatedly infinitely many times. Let $\omega_k$ denote the outcome on the $k^{\text{th}}$ toss. Denote be $\mathcal{F}_k$ the $\sigma$-algebra generated by the first $k$ tosses.

Define a random walk $S_0, S_1, S_2, \ldots$ in the following way. Set $S_0 = 0$, and then, recursively,

$$S_{k+1}(\omega) = \begin{cases} S_k(\omega) + 1 & \text{if } \omega_{k+1} = H, \\ S_k(\omega) - 1 & \text{if } \omega_{k+1} = T. \end{cases}$$

**(i)** Show that $S_0, S_1, S_2, \ldots$ is a martingale with respect to the filtration $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots$.

**(ii)** Let $\Delta_0, \Delta_1, \Delta_2, \ldots$ be another stochastic process adapted to the filtration $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots$. Define $I_0 = 0$, and for each positive integers $n$, set

$$I_n = \sum_{k=0}^{n-1} \Delta_k (S_{k+1} - S_k).$$

Show that $I_0, I_1, I_2, \ldots$ is adapted to the filtration $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots$ Is it a martingale?

(The process $I_0, I_1, I_2, \ldots$ is a discrete analogue of the *Itô integral.*)

**Problem 2.** (Generalized binomial model.)
Let $\Omega_N$ be the set of sequences of $N$ coin tosses, and let $\mathcal{F}_n$ be the $\sigma$-algebra of information generated by the first $n$ tosses, $n = 0, 1, \ldots, N$. Let $\{S_n\}_{n=0}^{N}$ be an adapted process of positive random variables, which represent stock prices. Let $\{r_n\}_{n=1}^{N-1}$ be an adapted process of nonegative random variables which represent interest rates, i.e., $r_n(\omega_1 \ldots \omega_n)$ is the interest rate paid by the money market between times $n$ and $n+1$ if $\omega_1 \ldots \omega_n$ is the outcome of the first $n$ coin tosses.

**(i)** Let $\{\Delta_n\}_{n=0}^{N-1}$ be an adapted sequence of random variables, where each $\Delta_n$ represents the number of shares of stock held between times $n$ and $n+1$. Let $X_0$ be nonrandom and represent the initial value of an agent's portfolio. In terms of $X_n$, the value of the agent's portfolio at time $n$, write a formula for $X_{n+1}$, the value of the agent's portfolio at time $n+1$.

**(ii)** Show that the process $\{X_n\}$ is adapted.

**(iii)** What is the extension of the Theorem on page 28 of lecture 7 for this case? Note that $\tilde{p}$ and $\tilde{q}$ must now depend on time and coin tosses, i.e., they are also stochastic processes. Are these processes adapted?

**(iv)** In a two-period model with $S_0 = 4$, $S_1(H) = 8$, $S_1(T) = 2$, $S_2(HH) = 10$, $S_2(HT) = 2$, $S_2(TH) = 5$, $S_2(TT) = 1$, $r_0 = \frac{1}{4}$, $r_1(H) = 0$, and $r_1(T) = \frac{1}{3}$, consider a European call expiring at time two and having strike price $K = 4$.

- **(a)** What are the time-one values $V_1(H)$ and $V_1(T)$ and the time-zero value $V_0$ of this call?
- **(b)** What are the values of $\Delta_0$, $\Delta_1(H)$ and $\Delta_1(T)$ for the portfolio which replicates this call?