A Card Trick

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1 Introduction

Suppose we have a deck of 21 cards that have been arranged in an array of seven rows and three columns. We ask a friend to choose a card and to tell you only which column contains the chosen card. For example, suppose that the initial array looks as follows:

$$
\begin{array}{ccc}
L_1 & C_1 & R_1 \\
L_2 & C_2 & R_2 \\
L_3 & C_3 & R_3 \\
L_4 & C_4 & R_4 \\
L_5 & C_5 & R_5 \\
L_6 & C_6 & R_6 \\
L_7 & C_7 & R_7
\end{array}
$$

If the friend chose R_6, she would then let us know that the card is in column 3. We then gather up the cards, one column at the time, with column 3 in the middle. We then lay down the cards in a 7 by 3 array, a row at a time, so our array would look as follows:

$$
\begin{array}{ccc}
L_1 & L_2 & L_3 \\
L_4 & L_5 & L_6 \\
L_7 & R_1 & R_2 \\
R_3 & R_4 & R_5 \\
R_6 & R_7 & C_1 \\
C_2 & C_4 & C_5 \\
C_5 & C_6 & C_7
\end{array}
$$

We again ask our friend to tell us which column contains the chosen card. She tells us it is now in column 1, so we repeat the process and obtain the following array:

$$
\begin{array}{ccc}
L_2 & L_5 & R_1 \\
R_4 & R_7 & C_4 \\
C_6 & L_1 & L_4 \\
L_7 & R_3 & R_6 \\
C_2 & C_5 & L_3 \\
L_6 & R_2 & R_5 \\
C_1 & C_5 & C_7
\end{array}
$$

If we now ask her which column contains her card, she will tell us column 3. At this point you can identify her card since it will be in the fourth row. Amir-Moéz showed in [1] for any set of $3q$ cards arranged in a rectangular array of q rows and 3 columns, where q is odd, the above algorithm will always eventually drive the chosen card to the middle position of its column. In this article we examine some generalizations of this card trick.

Consider a set of mq cards arranged in a rectangular array of q rows and m columns. We assume $m = 2h + 1$ is odd. Each time our friend identifies the column of her chosen card, we pick up the cards, column by column, from top to bottom, with the chosen column sandwiched between the other $2h$ columns. We can imagine first permuting the columns so that the chosen one is in the middle, then labelling the cards as follows: the cards in the h left-most columns are labelled L_1, L_2, …, L_{hq}; the cards in the h right-most columns are labelled R_1, R_2, …, R_{hq}; and the cards in the chosen column C_1, C_2, …, C_q. We then lay down the cards in another q by m array, now proceeding row by row. So for example, if $m = 5$ and $q = 7$, we have

$$\begin{array}{ccccc} L_1 & L_8 & C_1 & R_1 & R_8 \\ L_2 & L_9 & C_2 & R_2 & R_9 \\ L_3 & L_{10} & C_3 & R_3 & R_{10} \\ L_4 & L_{11} & C_4 & R_4 & R_{11} \\ L_5 & L_{12} & C_5 & R_5 & R_{12} \\ L_6 & L_{13} & C_6 & R_6 & R_{13} \\ L_7 & L_{14} & C_7 & R_7 & R_{14} \end{array} \Rightarrow \begin{array}{ccccc} L_1 & L_2 & L_3 & L_4 & L_5 \\ L_6 & L_7 & L_8 & L_9 & L_{10} \\ L_{11} & L_{12} & L_{13} & L_{14} & C_1 \\ C_2 & C_3 & C_4 & C_5 & C_6 \\ C_7 & R_1 & R_2 & R_3 & R_4 \\ R_5 & R_6 & R_7 & R_8 & R_9 \\ R_{10} & R_{11} & R_{12} & R_{13} & R_{14} \end{array}$$

So if the chosen card was originally in the third row (meaning that its label is C_3), it has now moved to the fourth row. Define $f(k)$ to be the row containing the chosen card after one iteration of this algorithm, given that the chosen card was initially in row k. So with $m = 5$ and $q = 7$, we have that $f(2) = f(3) = f(4) = f(5) = f(6) = 4$, $f(1) = 3$, and $f(7) = 5$. Notice that 4 is a fixed point of f, so once the chosen card is put in row 4, it stays there. We also note that $f^2(1) = f(f(1)) = f(3) = 4$ and $f^2(7) = f(f(7)) = f(5) = 4$. So in at most 2 iterations, any chosen card will be put into the fourth row. Therefore, once our friend identifies the column, we can identify the card.

We now consider the general case with q rows and $m = 2h + 1$ columns. Suppose the chosen card is in row k. Then C_k is the chosen card, and it is the $hq + k$th card to be laid down in the next iteration. Since each row has m cards, the row that will contain C_k is $\left\lceil \frac{hq+k}{m} \right\rceil$ where $\lceil x \rceil$ is the ceiling function. So we have that $f(k) = \left\lceil \frac{hq+k}{m} \right\rceil$. It follows that

$$\frac{hq + k}{m} \leq f(k) \leq \frac{hq + k}{m} + 1$$

for $1 \leq k \leq q$.

References

[1] Ali R. Amir-Moéz. Limit of a Function and a Card Trick *Mathematics Magazine,* **00** (1965), 191–196.