Trente Et Quarante

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Introduction

Although not commonly found in the United States, Trente et Quarante is very popular in French casinos. The game is played with six standard decks of cards. Each card is assigned a value equal to its rank (Ace is 1), with face cards (Jacks, Queens, and Kings) assigned a value of 10. The dealer, who represents the house, deals a row of cards, stopping when the sum of the values of the cards dealt first exceeds 30. This row is designated as Noir. A second row, Rouge, is then dealt in the same way. Thus each of the row totals will have a value in the set $\{31, 32, 33, \ldots, 40\}$. The row with a cumulative total closest to 30 wins. Before the game begins, players may make wagers on either Noir or Rouge. In addition, a player may bet on Couleur or Inverse. A bet on Couleur wins if the Noir row wins and the first card in the Noir row is black, or if the Rouge row wins and the first card in the Rouge row is red. Conversely, a bet on Inverse wins if the Noir row wins and the first card in the Noir row is red, or if the Rouge row wins and the first card in the Rouge row is black. It is easily seen that a all four bets have the same probability of winning.

If the two row totals are equal, the bet is a tie and nobody wins. However, if the two totals both equal 31, the house pockets half of all bets. We are interested in computing the expectation of this game.

We denote by $P(n)$ the probability that a row total will equal n. We wish to compute $P(n)$ for $n = 31, 32, 33, \ldots 40$. To simplify our calculations we assume sampling with replacement, so that the probability

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of dealing a card with value k is $\frac{1}{13}$ for $k = 1, 2, 3, \ldots, 9$, and the probability of dealing a card with value 10 is $\frac{4}{13}$.

We consider now the possible outcomes for the Noir row. Since we stop dealing cards as soon as the total exceeds 30, we may think of an outcome as an ordered sequence $(a_1, a_2, \ldots, a_l)$ of l card values such that $a_1 + a_2 + \ldots + a_{l-1} \leq 30$. Let $T(n, k, l)$ be the total number of these sequences of length l, where exactly k entries have value 10, and the sum of all entries is n. Although we are mainly interested in computing $T(n, k, l)$ for $n \geq 31$, it is convenient to allow n to be any integer, where we set $T(n, k, l) = 0$ for $n \leq 0$. So for example $T(13, 1, 3) = 6$, since there are six ways of achieving a row total of 13 with one face card and two non-face cards. These are $(10, 1, 2)$, $(10, 2, 1)$, $(1, 10, 2)$, $(2, 10, 1)$, $(1, 2, 10)$, and $(2, 1, 10)$.

Our goal is to compute $T(n, k, l)$ for $31 \leq n \leq 40$, $0 \leq k \leq 4$, and $k \leq l \leq n$, since then we can determine $P(n)$ by the formula

$$P(n) = \sum_{k=0}^{4} \sum_{l=k}^{40} \left(\frac{1}{13}\right)^{l-k} \cdot \left(\frac{4}{13}\right)^{k} \cdot T(n, k, l). \tag{1}$$

We will actually compute $T(n, k, l)$ for all $(n, k, l) \in A \times \{0, 1, 2, 3, 4\} \times A$, where $A = \{1, 2, 3, \ldots, 40\}$.

Computing $T(n, 0, l)$

First consider the case where $k = 0$, so that no face cards are dealt. Clearly

$$T(n, 0, 1) = \begin{cases} 1, & \text{for } n = 1, 2, 3, 4, \ldots, 9 \\ 0, & \text{otherwise} \end{cases}. \tag{2}$$

Now to compute $T(n, 0, l)$ for $l \geq 2$, we need to count the total number of sequences $(a_1, a_2, \ldots, a_l)$ with entries in $\{1, 2, 3, \ldots, 9\}$ such that $a_1 + a_2 + \ldots + a_{l-1} \leq 30$ and $a_1 + a_2 + \ldots a_l = n$. Assume first that $1 \leq n \leq 31$. The number of such sequences for which $a_l = 1$ is $T(n-1, 0, l-1)$, since if the last card has value 1, the remaining $l-1$ cards must add up to $n-1$ and contain no face cards. Similarly, the number of sequences for which $a_l = m$ is $T(n-m, 0, l-1)$ for $m = 1, 2, 3, \ldots 9$. Hence, for $1 \leq n \leq 31$ we have

$$T(n,0,l) = \sum_{m=1}^{9} T(n-m,0,l-1). \tag{3}$$

If $n = 32$, the situation is slightly different since the last card cannot have value 1 (otherwise that last card would not have been dealt since a total of 31 would have been reached on the $(l-1)$st card). Since the last card may assume any of the other values $2, 3, \ldots, 9$, we see that $T(32,0,l) = \sum_{m=2}^{9} T(32-m,0,l-1)$. More generally, for $32 \leq n \leq 40$,

$$T(n,0,l) = \sum_{m=n-30}^{9} T(n-m,0,l-1). \tag{4}$$

These recursive formulas can be used to compute $T(n,0,l)$ for all $(l,n) \in A \times A$.

One way to compute these is to set up an Excel spreadsheet. The formulas (3) and (4) tell us that the entry $T(n,0,l)$ in column n and row l can be obtained by summing entries from the previous row $l-1$. The entries in the first row are given by (2). The results for $n \geq 31$ are given below:

n	31	32	33	34	35	36	37	38	39	40
T(n,0,1)	0	0	0	0	0	0	0	0	0	0
T(n,0,2)	0	0	0	0	0	0	0	0	0	0
T(n,0,3)	0	0	0	0	0	0	0	0	0	0
T(n,0,4)	56	35	20	10	4	1	0	0	0	0
T(n,0,5)	2,430	1,974	1,554	1,179	855	585	369	204	84	0
T(n,0,6)	32,292	28,782	25,032	21,132	17,181	13,281	9,531	6,021	2,826	0
T(n,0,7)	233,331	217,734	199,122	177,470	152,894	125,657	96,165	64,953	32,661	0
T(n,0,8)	1,127,736	1,079,940	1,017,339	937,413	837,831	716,640	572,454	404,628	213,402	0
T(n,0,9)	4,039,335	3,929,391	3,772,575	3,554,874	3,260,250	2,871,042	2,368,602	1,734,156	949,860	0
T(n,0,10)	11,377,750	11,178,715	10,870,528	10,407,241	9,729,685	8,763,940	7,420,419	5,593,842	3,164,355	0
T(n,0,11)	26,168,769	25,877,039	25,386,769	24,589,599	23,332,145	21,403,570	18,522,120	14,321,450	8,337,880	0
T(n,0,12)	50,395,500	50,043,510	49,400,010	48,266,955	46,338,732	43,158,060	38,060,253	30,105,306	17,998,299	0
T(n,0,13)	82,672,213	82,319,641	81,615,145	80,267,435	77,787,671	73,382,687	65,805,039	53,149,032	32,581,836	0
T(n,0,14)	116,910,990	116,617,073	115,970,596	114,619,701	111,921,460	106,744,820	97,167,564	80,025,036	50,258,325	0
T(n,0,15)	143,678,475	143,474,985	142,977,579	141,833,709	139,339,035	134,146,575	123,779,295	103,840,947	66,778,635	0
T(n,0,16)	154,249,296	154,133,016	153,813,246	152,996,071	151,035,040	146,579,440	136,931,940	116,919,540	76,977,360	0
T(n,0,17)	145,076,742	145,022,478	144,851,934	144,361,620	143,054,132	139,785,628	132,061,644	114,690,840	77,310,696	0
T(n,0,18)	119,652,120	119,631,771	119,557,158	119,312,001	118,576,530	116,533,572	111,222,126	98,186,832	67,781,550	0
T(n,0,19)	86,467,955	86,461,970	86,435,636	86,334,689	85,988,585	84,907,010	81,782,478	73,346,517	51,875,415	0
T(n,0,20)	54,623,100	54,621,770	54,614,455	54,580,806	54,446,210	53,965,510	52,403,235	47,716,429	34,593,680	0
T(n,0,21)	30,044,574	30,044,364	30,042,824	30,033,969	29,991,465	29,814,365	29,156,565	26,936,490	20,029,610	0
T(n,0,22)	14,307,128	14,307,107	14,306,876	14,305,105	14,294,479	14,241,349	14,011,119	13,123,089	10,014,984	0
T(n,0,23)	5,852,925	5,852,924	5,852,902	5,852,649	5,850,625	5,837,975	5,772,195	5,476,185	4,292,145	0
T(n,0,24)	2,035,800	2,035,800	2,035,799	2,035,776	2,035,500	2,033,200	2,018,250	1,937,520	1,560,780	0
T(n,0,25)	593,775	593,775	593,775	593,774	593,750	593,450	590,850	573,300	475,020	0
T(n,0,26)	142,506	142,506	142,506	142,506	142,505	142,480	142,155	139,230	118,755	0
T(n,0,27)	27,405	27,405	27,405	27,405	27,405	27,404	27,378	27,027	23,751	0
T(n,0,28)	4,060	4,060	4,060	4,060	4,060	4,060	4,059	4,032	3,654	0
T(n,0,29)	435	435	435	435	435	435	435	434	406	0
T(n,0,30)	30	30	30	30	30	30	30	30	29	0
T(n,0,31)	1	1	1	1	1	1	1	1	1	0
T(n,0,32)	0	0	0	0	0	0	0	0	0	0
T(n,0,33)	0	0	0	0	0	0	0	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
T(n,0,40)	0	0	0	0	0	0	0	0	0	0

Table 1: $T(n, 0, l)$ is the number of sequences with l cards, no face cards, with the values adding up to n, and such that the total of the first $l - 1$ cards is ≤ 30

Computing $T(n, k, l)$ for $k = 1, 2, 3, 4$

To compute $T(n, k, l)$ when $k \geq 1$ we must take into account the possibility that the last card has value 10, in which case the preceding $l - 1$ cards must add up to $n - 10$ using $k - 1$ 10's. Thus the formulas are

$$T(n,k,l) = \begin{cases} \displaystyle\sum_{m=1}^{9} T(n-m,k,l-1) + T(n-10,k-1,l-1), & \text{if } n \leq 31 \\ \displaystyle\sum_{m=n-30}^{9} T(n-m,k,l-1) + T(n-10,k-1,l-1), & \text{if } n \geq 32 \end{cases}.$$

Again we can compute these using Excel. The first row of each table is given by the following:

$$T(n,k,k) = \begin{cases} 1 & \text{if } n = 10k \\ 0 & \text{otherwise} \end{cases}$$

for $k = 1, 2, 3, 4$. So for example, each entry $T(n, 1, l)$ where $l \geq 2$ in Table 2 below can be computed by summing entries from the previous row of Table 2 and one entry from Table 1. Again, we only give here the entries corresponding to $31 \leq n \leq 40$.

n	31	32	33	34	35	36	37	38	39	40
T(n,1,1)	0	0	0	0	0	0	0	0	0	0
T(n,1,2)	0	0	0	0	0	0	0	0	0	0
T(n,1,3)	0	0	0	0	0	0	0	0	0	0
T(n,1,4)	112	84	60	40	24	12	4	0	0	0
T(n,1,5)	2,400	2,168	1,904	1,619	1,324	1,030	748	489	264	84
T(n,1,6)	19,170	18,660	17,820	16,620	15,051	13,125	10,875	8,355	5,640	2,826
T(n,1,7)	89,124	90,009	90,054	88,834	85,902	80,823	73,208	62,748	49,248	32,661
T(n,1,8)	284,208	293,244	302,505	310,821	316,505	317,345	310,646	293,322	262,038	213,402
T(n,1,9)	673,920	705,288	744,768	791,925	844,824	899,424	948,984	983,550	989,592	949,860
T(n,1,10)	1,244,850	1,316,430	1,418,454	1,558,110	1,741,491	1,971,765	2,246,653	2,555,253	2,874,339	3,164,355
T(n,1,11)	1,841,510	1,964,180	2,157,770	2,451,800	2,882,054	3,488,825	4,313,000	5,389,210	6,735,340	8,337,880
T(n,1,12)	2,215,620	2,382,370	2,671,460	3,153,150	3,926,296	5,123,690	6,914,985	9,504,715	13,122,065	17,998,299
T(n,1,13)	2,183,324	2,367,816	2,718,948	3,358,730	4,478,772	6,367,956	9,444,524	14,289,507	21,675,600	32,581,836
T(n,1,14)	1,763,580	1,931,514	2,283,918	2,987,322	4,329,936	6,790,590	11,134,422	18,538,806	30,749,277	50,258,325
T(n,1,15)	1,162,800	1,288,770	1,582,672	2,228,954	3,578,484	6,269,900	11,419,240	20,903,676	37,767,744	66,778,635
T(n,1,16)	620,160	697,680	901,170	1,398,560	2,542,206	5,035,200	10,218,700	20,547,900	40,349,160	76,977,360
T(n,1,17)	263,568	302,328	418,608	738,378	1,555,536	3,516,312	7,969,872	17,605,812	37,566,192	77,310,696
T(n,1,18)	87,210	102,714	156,978	327,522	817,836	2,125,306	5,393,522	13,115,058	30,471,174	67,781,550
T(n,1,19)	21,660	26,505	46,854	121,467	366,624	1,102,095	3,145,034	8,456,157	21,488,544	51,875,415
T(n,1,20)	3,800	4,940	10,925	37,259	138,206	484,310	1,565,885	4,690,397	13,125,998	34,593,680
T(n,1,21)	420	610	1,940	9,255	42,904	177,500	658,200	2,220,475	6,907,260	20,029,610
T(n,1,22)	22	42	252	1,792	10,647	53,151	230,251	888,051	3,108,126	10,014,984
T(n,1,23)	0	1	22	253	2,024	12,650	65,780	296,010	1,184,040	4,292,145
T(n,1,24)	0	0	1	23	276	2,300	14,950	80,730	376,740	1,560,780
T(n,1,25)	0	0	0	1	24	300	2,600	17,550	98,280	475,020
T(n,1,26)	0	0	0	0	1	25	325	2,925	20,475	118,755
T(n,1,27)	0	0	0	0	0	1	26	351	3,276	23,751
T(n,1,28)	0	0	0	0	0	0	1	27	378	3,654
T(n,1,29)	0	0	0	0	0	0	0	1	28	406
T(n,1,30)	0	0	0	0	0	0	0	0	1	29
T(n,1,31)	0	0	0	0	0	0	0	0	0	1
T(n,1,32)	0	0	0	0	0	0	0	0	0	0
T(n,1,33)	0	0	0	0	0	0	0	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
T(n,1,40)	0	0	0	0	0	0	0	0	0	0

Table 2: $T(n,1,l)$ is the number of sequences with 1 face card and $l-1$ non-face cards, with the values adding up to n, and such that the total of the first $l-1$ cards is ≤ 30

The results for $k=2$ and $k=3$ and $n \geq 31$ are given below in Tables 3 and 4:

n	31	32	33	34	35	36	37	38	39	40
T(n,2,2)	0	0	0	0	0	0	0	0	0	0
T(n,2,3)	0	0	0	0	0	0	0	0	0	0
T(n,2,4)	48	42	36	30	24	18	12	6	0	0
T(n,2,5)	450	472	480	474	454	420	372	310	234	144
T(n,2,6)	1,800	2,025	2,270	2,510	2,720	2,875	2,950	2,920	2,760	2,445
T(n,2,7)	4,410	5,130	6,120	7,395	8,931	10,665	12,495	14,280	15,840	16,956
T(n,2,8)	7,056	8,526	10,836	14,301	19,243	25,935	34,545	45,080	57,330	70,812
T(n,2,9)	7,560	9,576	13,272	19,608	29,904	45,836	69,356	102,536	147,336	205,296
T(n,2,10)	5,400	7,290	11,448	19,764	35,208	62,235	107,172	178,488	286,848	444,852
T(n,2,11)	2,475	3,675	6,975	14,895	32,055	66,375	130,725	244,990	438,390	751,770
T(n,2,12)	660	1,155	2,970	8,415	22,572	55,605	126,390	267,960	535,205	1,015,058
T(n,2,13)	78	198	858	3,498	12,078	36,102	96,162	233,442	525,162	1,108,404
T(n,2,14)	0	13	156	1,014	4,732	17,745	56,784	160,888	413,712	982,566
T(n,2,15)	0	0	14	182	1,274	6,370	25,480	86,632	259,896	705,432
T(n,2,16)	0	0	0	15	210	1,575	8,400	35,700	128,520	406,980
T(n,2,17)	0	0	0	0	16	240	1,920	10,880	48,960	186,048
T(n,2,18)	0	0	0	0	0	17	272	2,312	13,872	65,892
T(n,2,19)	0	0	0	0	0	0	18	306	2,754	17,442
T(n,2,20)	0	0	0	0	0	0	0	19	342	3,249
T(n,2,21)	0	0	0	0	0	0	0	0	20	380
T(n,2,22)	0	0	0	0	0	0	0	0	0	21
T(n,2,23)	0	0	0	0	0	0	0	0	0	0
T(n,2,24)	0	0	0	0	0	0	0	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
T(n,2,40)	0	0	0	0	0	0	0	0	0	0

Table 3: $T(n, 2, l)$ is the number of sequences with 2 face cards and $l - 2$ non-face cards, with values adding up to n, and such that the total of the first $l - 1$ cards is ≤ 30

n	31	32	33	34	35	36	37	38	39	40
T(n,3,3)	0	0	0	0	0	0	0	0	0	0
T(n,3,4)	4	4	4	4	4	4	4	4	4	0
T(n,3,5)	0	6	12	18	24	30	36	42	48	54
T(n,3,6)	0	0	10	30	60	100	150	210	280	360
T(n,3,7)	0	0	0	15	60	150	300	525	840	1,260
T(n,3,8)	0	0	0	0	21	105	315	735	1,470	2,646
T(n,3,9)	0	0	0	0	0	28	168	588	1,568	3,528
T(n,3,10)	0	0	0	0	0	0	36	252	1,008	3,024
T(n,3,11)	0	0	0	0	0	0	0	45	360	1,620
T(n,3,12)	0	0	0	0	0	0	0	0	55	495
T(n,3,13)	0	0	0	0	0	0	0	0	0	66
T(n,3,14)	0	0	0	0	0	0	0	0	0	0
T(n,3,15)	0	0	0	0	0	0	0	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
T(n,3,40)	0	0	0	0	0	0	0	0	0	0

Table 4: $T(n, 3, l)$ is the number of sequences with 3 face cards and $l - 3$ non-face cards, with values adding up to n, and such that the total of the first $l - 1$ cards is ≤ 30

Finally, if $k = 4$, we have that $T(n, 4, l) = \begin{cases} 1, & \text{if } n = 40 \text{ and } l = 4 \\ 0, & \text{otherwise} \end{cases}$.

Computing the Expected Value of Trente et Quarante

Having computed all the entries in the above tables, we can now compute $P(n)$ for $n = 31, 32, \ldots, 40$ using formula (1). We obtain the following

n	31	32	33	34	35	36	37	38	39	40
$P(n)$	0.14806	0.1379052	0.1275127	0.1168911	0.1060495	0.0949984	0.0837498	0.0723173	0.0607161	0.0517991

These probabilities were first computed by Denis Poisson [2]. Our method of computation makes use of technological tools unavailable in the nineteenth century.

Having computed the probabilities of each row total, we can now find the expected value of this game. The probability that the two rows Rouge and Noir will have the same sum is

$$P(\text{tie}) = [P(31)]^2 + [P(32)]^2 + [P(33)]^2 + \ldots + [P(40)]^2 = 0.1097475.$$

Now since $P(\text{Rouge wins}) = P(\text{Noir wins})$ and $P(\text{Rouge wins}) + P(\text{Noire wins}) + P(\text{tie}) = 1$, we can compute

$$P(\text{Rouge wins}) = \frac{1 - P(\text{tie})}{2} = \frac{1 - 0.109744}{2} = 0.445128.$$

Finally, recalling that the house collects half of all bets if there's a tie at 31, we can compute E, the expected winnings per dollar bet on Rouge:

$$E = 1 \times P(\text{Rouge wins}) - 1 \times P(\text{Noire wins}) - \frac{1}{2} \times [P(31)]^2 = -.01095.$$

Thus, on average a player will lose approximately 1.1 cent for every dollar bet.

References

[1] Richard A. Epstein. *The Theory of Gambling and Statistical Logic*, Academic Press, 1977.

[2] Denis Poisson. “Mémoire sur L’Avantage du Banquier au Jeu de Trente et Quarant,” *Ann. Mat. Pures Appl.* **xvi**, N. vi (Decembre, 1825) pp. 173–208.

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