# Can galaxies exist within our particle horizon with Hubble recessional velocities greater than $c$?

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A simple, global picture of photon exchange in the flat, matter-dominated universe is presented using relativistic cosmology. Equations for the photon's recessional velocity and position relative to the receiver are derived. The results are discussed in the context of the "raisin bread universe." Contrary to intuition, it is shown that the Hubble recessional velocity of the emitter can exceed the speed of light for sources within the particle horizon of the receiver. The model used to obtain these results can be used by introductory astronomy students to obtain the proper distance and the time-of-flight distance to sources with large redshifts.

## I. INTRODUCTION

It is widely accepted among astronomers that general relativistic cosmology allows for Hubble recessional velocities greater than the speed of light $c$. However, it is not well known that some of these sources may lie within our particle horizon. In the explanation of this result, it is convenient (and at times necessary) to introduce concepts such as redshift, time-of-flight distance, proper distance, cosmic microwave background, curved space-time, Hubble's law, and particle horizon. These are concepts associated with the fields of general relativity (GR), astronomy, and cosmology, but this paper assumes only that the reader is familiar with calculus and special relativity. Therefore, this presentation begins with two sections which introduce the necessary concepts of astronomy, GR, and cosmology.

The necessary terminology is explained in Sec. II. Terms such as particle horizon, cosmic microwave background (CMB), cosmological expansion, and redshift are explained there. General relativity is also discussed conceptually in Sec. II. An explanation of Hubble's law and its use in astronomical distance measurements follows in Sec. III. The reader who is already familiar with these concepts can skip Secs. II and III without loss of continuity.

Section IV sets forth all the necessary mathematics of the paper. The section begins with a description of the mathematical model. Following a description of the mathematical model, several pertinent equations are derived. These equations (valid for use on very distant objects) describe a global picture of photon exchange in the flat, matter-dominated universe of GR. These results are expressed predominantly in terms of proper distance and proper time. However, astronomers typically use time-of-flight (TOF) distance when discussing very distant objects, thus TOF distance is also addressed. The reader uninterested in the mathematical details can skim Sec. IV for definitions. Knowledge of the definitions in Sec. IV will suffice for a qualitative comprehension of the example (Sec. V) and discussion (Sec. VI).

## II. INTRODUCTORY CONCEPTS

This paper adopts, *a priori*, the fact that the universe is expanding. The material contents of the universe are not expanding into space, but rather space itself is expanding, carrying along with it the content of the universe. Many books use illustrations of receding galaxies, raisin bread, and balloons to depict the concept of cosmological expansion.[1] In these representations the reader is privy to a view from "outside" the universe. This point of view is referred to as the "raisin bread universe." The raisins represent galaxies and the expanding bread represents space.

Tracing the cosmological expansion retrotemporally, one concludes that at one time the contents of the universe existed in a state of infinite density and infinite temperature. This initial singularity is called the Big Bang. The existence of an initial singularity is an inevitable result of GR cosmologies of physical interest.[2]

General relativity is a generalization of special relativity (SR) in that GR allows for relative acceleration between inertial reference frames. As applied to cosmology, GR is a theory of gravity. Simply put, matter and energy curve space-time and objects in freefall have worldlines of extremal length, called geodesics, in the curved space-time. SR holds in the inertial frames of GR, i.e., locally about each inertial observer. Of particular interest in GR cosmology is the set of inertial frames which detect an isotropic cosmic microwave background.[3]

The CMB is the radiation freed from scattering after the free electrons and protons combined to form hydrogen. This is often referred to as the "recombination epoch" and occurred after the universe had cooled (due to its expansion) to below about 4000 K. Prior to the recombination epoch, the photons (distributed according to Planck's law), had sufficient energy to ionize the hydrogen. After the formation of hydrogen, the photons were effectively decoupled from matter. The cosmological expansion has forced these photons' wavelengths to expand, thereby cooling them to their present temperature of 2.73 K. This change in wavelength due to the cosmological expansion is called cosmological redshift.

Quantitatively, redshift is defined as the change in wavelength (wavelength received minus wavelength emitted) divided by the emitted wavelength.[4] In this paper it is assumed that large redshifts are due predominantly to the cosmological expansion.[5] Thus the larger the redshift, the longer the photon has been in flight, and the farther its emitter is from us. The age of the universe is finite, therefore, there is a distance limit to what we can observe in the universe. That limit is called the particle horizon.[6]

Although the particle horizon is the boundary of our observable universe, we do not receive light from the particle horizon for two reasons. First, the redshift of a photon emitted at the Big Bang is infinite by definition. A photon with an infinite wavelength has no energy and cannot be detected. Second, the universe was opaque until hydrogen recombination. Therefore, the oldest photons we can detect are those of the CMB which correspond to a redshift of about 1500. Redshift is typically converted to distance via Hubble's law.

## III. HUBBLE'S LAW AND DISTANCE

In 1929, Edwin Hubble announced that a galaxy's recessional velocity was approximately proportional to its distance from us.[7] The mathematical statement $v = H_0 r$ is now known as Hubble's law, where $v$ is the Hubble recessional velocity, $r$ is the distance between us and the galaxy in question, and $H_0$ is called the Hubble constant. For $v \ll c$ the approximate equation $v = zc$ is valid, where $z$ is the redshift.

If $H_0$ is known, the small $z$ form of Hubble's law, $zc = H_0 r$, can be used to find distance in the range $0.003 \lesssim z \lesssim 0.5$.[8] The lower limit might be explained as the point where peculiar galaxy velocity,[9] i.e., motion not associated with the cosmological expansion, is of the order of the cosmological expansion velocity. The upper limit exists because of the cosmological dependence of $r(z)$.

That is to say, the function $r$ of $z$ depends on which cosmology model is employed. GR cosmology plus the cosmological principle, i.e., the assumption that the universe looks isotropic in the large at all points in space, yield three possible cosmology models. In these models the universe has either zero spatial curvature (flat), positive spatial curvature (spherical), or negative spatial curvature (hyperboloid). For $z \lesssim 0.5$ these models' $r(z)$ reduce to the small $z$ Hubble's law.[10] For $z \gtrsim 0.5$ a model must be chosen and the type of distance specified.[11]

There are many different concepts of distance used in astronomy and cosmology.[12] These distances, essentially equal for small $z$, differ widely for large $z$. Astronomers typically use the time-of-flight (TOF) distance when referring to large $z$ objects.[13] TOF distance is the time it takes a photon to travel between emitter and receiver multiplied by $c$. The distance that fits intuitively with a raisin bread universe is that of proper distance. The proper distance is the spatial separation between emitter and receiver that exists at any given time. Because TOF distance is widely referenced and proper distance fits the intuitive raisin bread universe, both forms are used in equations which are derived in the next section. TOF distance and proper distance are model dependent, but easily obtained in the flat, GR cosmology model.

## IV. THE MODEL

The cosmology model used in this paper is the flat, matter-dominated GR cosmology model. "Matter-dominated" means that pressure (e.g., radiation pressure) is assumed negligible and set equal to zero in the GR equations of motion. Because an understanding of the GR equations of motion requires a detailed knowledge of GR, some results herein must be stated without proof. (The omission of the GR dependent calculations is not critical to the derivations, but the interested reader can find the omitted details in any standard general relativity text.) Qualitative definitions of time and proper distance for this model are established before beginning the calculations.

The set of observers that detect an isotropic CMB (each member of the set is abbreviated "IO") is used to define the concepts of time and distance. The proper time measured by the IOs, synchronized so that $t = 0$ is the Big Bang, is what is meant by time. Therefore, the rising of the raisin bread represents billions of years (Gyrs). Galaxies are assumed to be IOs, consistent with the assumption that the cosmological recessional velocity dominates peculiar galactic velocities for large $z$. The instantaneous spatial separation between IOs, measured in billions of light years (Gcyrs), is what is meant by proper distance. Thus proper distance is the distance given by the metric on any space-time surface of constant time.

The metric $ds^2$ gives the infinitesimal distance measure of the space-time and therefore establishes the geometry of space. The metric that corresponds to the above concepts of distance and time for the spatially flat universe is

$$ds^2 = -c^2\,dt^2 + a^2(t)(d\chi^2 + \chi^2\,d\theta^2 + \chi^2 \sin^2\theta\,d\phi^2). \tag{1}$$

Obviously, the constant-time hypersurfaces are Euclidean (flat) spaces covered with spherical coordinates. The coordinates are affixed to the network of IOs and thus expand with the universe. Therefore, the coordinate distance between any pair of IOs is constant. The changing distance between the IOs is specified by $a(t)$. The time in this metric is the proper time as measured by the IOs. For the flat, matter-dominated universe general relativity gives

$$a(t) = \beta t^{2/3}, \tag{2}$$

where $\beta$ is a constant that will not affect the calculations. [This form of $a(t)$ is in disregard to any inflationary epochs that may have occurred in the early universe.[14] ]

For simplicity assign the receiver $\chi = 0$. The light will then proceed radially (i.e., with $\theta$ and $\phi$ constant) to the receiver. This gives the proper distance between emitter and receiver at time $t$ as

$$r_e(t) = \chi_e a(t) = \chi_e \beta t^{2/3}, \tag{3}$$

where $\chi_e$ is the radial coordinate of the emitter. This is an allowed spatial path for the photon provided that it travels along a null geodesic ($ds^2 = 0$), so,

$$a(t)d\chi = -c\,dt, \tag{4}$$

where the negative sign indicates that the photon is traveling toward smaller values of $\chi$. The recessional velocity of the source is defined as

$$v := \dot{r}_e = \chi_e \dot{a}, \quad \dot{} = \frac{d}{dt}. \tag{5}$$

Equation (5) can be rewritten as

$$v = r_e H, \tag{6}$$

where $H = \dot{a}/a$. Equation (6) is the relativistic form of Hubble's law, $H$ is the "Hubble constant," and $v$ is the Hubble recessional velocity. Using $a(t)$ from Eq. (2) the Hubble "constant" is

$$H = 2/(3t). \tag{7}$$

Equation (4) can be integrated using Eq. (2) to obtain

$$\chi_e = 3c(t_0^{1/3} - t_e^{1/3})/\beta, \tag{8}$$

where $t_e$ is the photon emission time, and $t_0$ is the photon reception time. Equations (3) and (8) give

$$r_e(t_0) = 3ct_0^{2/3}(t_0^{1/3} - t_e^{1/3}). \tag{9}$$

The cosmological redshift is given by

$$z \equiv \Delta\lambda/\lambda = [a(t_0)/a(t_E)] - 1 \tag{10}$$

or using Eq. (2)

$$z = (t_0^{2/3} - t_e^{2/3})/t_e^{2/3}. \tag{11}$$

Equations (7), (9), and (11) give

$$r_e(t_0) = 2c[1 - (1+z)^{-1/2}]/H_0, \tag{12}$$

where $H_0 = H(t_0)$. Using Eqs. (6) and (12) gives the Hubble recessional velocity of the source at time $t_0$ as

$$v(t_0) = H_0 r_e(t_0) = 2c[1 - (1+z)^{-1/2}]. \tag{13}$$

[Notice that we obtain $v(t_0) > c$ for $z > 3$.] Using Eq. (11), the time-of-flight distance for the photon takes the form

$$R_t \equiv c(t_0 - t_e) = ct_0[1 - (1+z)^{-3/2}] \tag{14}$$

and with Eq. (7)

$$R_t = 2c[1 - (1+z)^{-3/2}]/(3H_0). \tag{15}$$

The photon's proper distance from the receiver as a function of time can be obtained by integrating Eq. (4) to arbitrary time $t$ acquiring

$$r_p = \chi_p(t)a(t) = r_e - 3ct^{2/3}(t^{1/3} - t_e^{1/3}), \tag{16}$$

where, of course,

$$\chi_p(t_e) = \chi_e \tag{17}$$

and

$$\chi_p(t_0) = 0. \tag{18}$$

Using Eqs. (3) and (8), Eq. (16) can also be written

$$r_p = 3ct\,[(t_0/t)^{1/3} - 1]. \tag{19}$$

Defining a photon recessional velocity analogous to the Hubble recessional velocity of the source gives

$$\dot{r}_p = \dot{r}_e + 2c(t_e/t)^{1/3} - 3c \tag{20}$$

or equivalently

$$\dot{r}_p = 2c(t_0/t)^{1/3} - 3c. \tag{21}$$

The above derivations and formulas are well suited for a calculus-based introductory astronomy course.[15] Also, these results incorporate gravitational acceleration effects, whereas many introductory astronomy texts do not. I have found that students of trigonometry-based astronomy courses can use the results after being shown an example.

## V. EXAMPLE

The example employed here is that of a recently discovered quasar[16] with a redshift of 4.73. To use the above equations, a value of $H_0$ is needed. The value of $H_0$ used here is 52 (km/s)/Mpc (Ref. 17) (1 Mpc = 0.00326 Gcyrs). Equation (7) then gives $t_0 = 13$ Gyrs. This $t_0$ is a far cry from the 18 Gyrs needed to produce the oldest stars, but the uncertainties in both $H_0$ and the ages of the oldest stars render these numbers compatible.[18]

Continuing with the example, Eq. (11) gives $t_e = 0.95$ Gyrs. Equation (9) [or equivalently Eq. (12)] yields 22 Gcyrs as the current proper distance to the quasar. Using Eqs. (10) and (3) allows for the computation of the proper distance between us and the quasar at emission time. This distance was thus $(z+1)$ times smaller than the separation at time of reception, or 3.8 Gcyrs. The current recessional velocity of the quasar is 1.2$c$ from Eq. (13). The recessional velocity of the quasar at emission time can be found from combining Eqs. (5), (2), and (11) to obtain a coordinate free equation for $v(t_0)/v(t_e)$. This then gives the recessional velocity of the quasar at emission time as

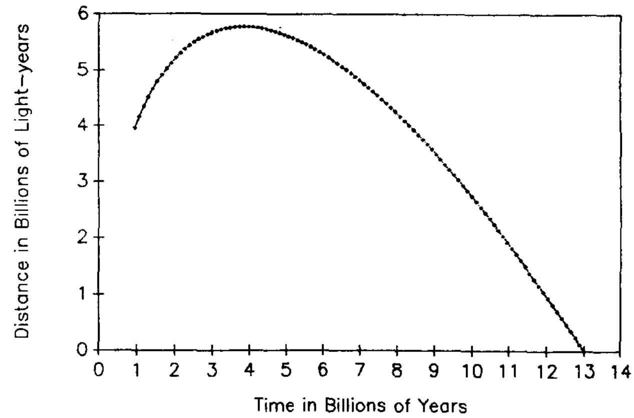


Fig. 1. The photon is emitted from a quasar at $t_e = 0.95$ Gyrs. The quasar (and thus the photon) is 3.8 Gcyrs away from us at time of emission. Initially the photon is "dragged away from us" by the cosmological expansion to a distance of 5.8 Gcyrs at $t = 3.9$ Gyrs. At this time gravity has slowed the expansion rate of space such that the photon is at a coordinate position with recessional velocity of $c$. The photon then begins to approach us and arrives at $t_0 = 13$ Gyrs.

2.8$c$ or $(z+1)^{1/2}$ times the current quasar recessional velocity.

The photon's proper distance at time of emission coincides with that of the quasar (as it must) as seen by Eq. (16) (Fig. 1). The distance of the photon at reception is, of course, zero as given by Eq. (19). The recessional velocity of the photon at time of emission is $\dot{r}_e - c$ from Eq. (20). Thus the photon is initially moving away from us at 1.8$c$ (Fig. 2). At time of reception the photon's recessional velocity is $-c$ as given by Eq. (21). These results follow Newtonian intuition which is verified by combining Eqs. (4), (16), and (20) to get[19]

$$\dot{r}_p = Hr_p - c. \tag{22}$$

The quasar's TOF distance is given by Eq. (14) as 12 Gcyrs. This is simply because the photon has traveled at speed $c$ for 12 billion years. The particle horizon is at a TOF distance of $ct_0$. [This is true of all models, but $t_0(H_0)$ is

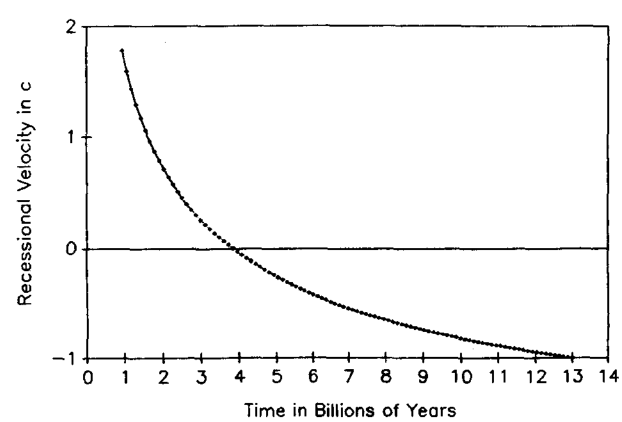


Fig. 2. Because the photon is being "dragged away from us" initially, its recessional velocity is initially positive. At emission the photon is receding at 1.8$c$. $\dot{r}_p = 0$ corresponds to the time when gravity has slowed the recessional velocity of space at the photon's position to $c$. After $t = 3.9$ Gyrs, the amount of expanding space between the photon and receiver is decreasing and the photon approaches the receiver at an increasing rate until the recessional velocity at reception is $-c$.

model dependent. Therefore, the TOF distance to the particle horizon, $r_H$, cannot be found by simply assigning $v = c$ in Hubble's law to get $r_H = c/H_0$.]

The above result which gives $v > c$ for the quasar within our particle horizon does not violate special relativity. Special relativity holds only locally in general relativity. These are for the most part global results. The one velocity which we measure locally (and the only velocity we can measure directly) is that of the photon at time of reception. This velocity has a magnitude of $c$ as derived above. However, these global relative motions are observable in the raisin bread universe.

## VI. DISCUSSION

The above process could have been described from the point of view of the quasar. In that case we (the Milky Way galaxy) were at a distance of 3.8 Gcyrs and moving away from the quasar at $2.8c$ when the photon was emitted. The Milky Way has been decelerating with respect to the source ever since. Currently we are 22 Gcyrs away from the quasar and receding at a velocity of $1.2c$ relative to the quasar. How is it that the photon, traveling at speed $c$ with respect to every observer along its path, can overtake us?

There are two ways to answer this question, mathematically and conceptually. Mathematically, begin with the requirement that the photon travel along a null geodesic [Eq. (4)]. In this equation notice that the photon will experience an infinitesimal coordinate displacement $d\chi$ with each lapse of infinitesimal proper time $dt$ for finite $a(t)$. The equations of GR, by dictating the function $a(t)$, then lead to a nonconvergent sum of the photon's $d\chi$'s [shown in Eq. (8)]. Thus the method of general relativistic cosmology used to describe the expansion of space (for this model) coupled with the requirement that the photon follow a null geodesic, guarantees the photon will traverse any coordinate separation given enough time.

Conceptually, the photon is swept away from us until gravity has slowed the expansion rate of space to the degree that the photon exists at a coordinate position with recessional velocity $c$. (Refer to Figs. 1 and 2 for the remainder of this section.) At this time the photon begins to approach us. In other words, the photon must overcome only the expansion effect that lies between it and the receiver.

The expansion of the universe must be viewed to occur uniformly throughout all of space. The mutual recessional velocity of any pair of IOs is a result of the entirety of expanding space that exists between them. Thus the expansion that occurs between the photon and the emitter will not affect the photon's motion toward the receiver. In fact, if all the expansion occurred "behind" the photon in the above example, then the photon would have reached our coordinate position in only 3.8 Gyrs. Instead, with the uniform expansion of space, the photon traveled for 12 Gyrs.

Therefore, the mutual recessional velocity between source and receiver at $t > t_e$ is immaterial to the photon's ability to traverse the space between them. The photon is "dragged away" initially, but always exists between the emitter and receiver because it moves through the coordinate system, whereas the emitter and receiver have fixed coordinates. When gravity has slowed the expansion rate of space so that the photon's coordinate location is receding at $c$, the photon begins to approach the receiver. It may be, as in the above example, that gravity does not slow the emitter's recessional velocity to $v < c$ before the photon arrives at the receiver, but that has nothing to do with the photon. (See also Harrison's explanation using the "Hubble sphere"[20] and Murdoch's paper[21] on the subject.)

## VII. CONCLUSIONS

First, in the flat matter-dominated cosmology model of general relativity, it is possible for us to receive photons from sources with Hubble recessional velocities greater than $c$. Special relativity is not compromised in this result because general relativity maintains SR only locally. In fact, calculations with this model elucidate special relativity's role in GR.

Second, the flat, matter-dominated GR cosmology model provides for derivations which can be presented in an introductory calculus-based astronomy course. Further, the formulas derived herein can be used by astronomy students at all levels to obtain TOF distance, recessional velocity, and proper distance for large $z$ objects. Thus the results are easily applicable and fully incorporate gravitational effects.


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# Quantum interference viewed in the framework of probability theory

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An additivity equality of probability theory is suggested to define (relative) noninterference. Interference is deviation from this additivity. It is shown that two-slit interference fits into this scheme. It is argued that the physical meaning of quantum interference is an observable deviation of quantum filtering from quantum occurrence of events. The expounded view should restore the essence of textbook treatment of interference in a precise and correct probability-theoretic framework. The approach is essentially a continuation of a critical argument presented by Ballentine.

## I. INTRODUCTION

Ballentine has challenged[1] (similarly as Koopman has done[2]) the usual imprecise probability-theoretic discussion of two-slit interference (that seems to go back to Feynman[3]), as well as that of general interference, claiming that it is erroneous. It seems to have remained unsettled if this is a question of detail or of principle, i.e., if interference can be understood within probability theory or not.

In this article it is shown that the essential content of the usual discussion of interference can be given a precise and correct probability-theoretic form (along the lines of Ballentine's argument).

We start with a sufficiently detailed (textbook) presentation of two-slit interference. Let the state vector $|\Psi(t)\rangle$ describe a beam of neutrons hitting at the instant $t>0$ a screen with two slits in it (numbered 1 and 2). Picturing the screen as an infinite one-dimensional $x$ axis $\mathbf{R}_1$, we have three disjoint regions $R_1 \equiv \{x: x \text{ in slit } 1\}$, $R_2 \equiv \{x: x \text{ in slit } 2\}$, and $R_0 \equiv \mathbf{R}_1 - (R_1 + R_2) = \{x: x \text{ in the screen}\}$.

Introducing the corresponding disjoint (or exclusive) quantum-mechanical events (orthogonal projectors)

$$P_1 \equiv \int_{R_1} dx|x\rangle\langle x|, \quad P_2 \equiv \int_{R_2} dx|x\rangle\langle x|,$$

and

$$P_0 \equiv \int_{R_0} dx|x\rangle\langle x|,$$

one has $P_1 + P_2 + P_0 = 1$, and, in terms of state vectors,

$$|\Psi(t)\rangle = P_1|\Psi(t)\rangle + P_2|\Psi(t)\rangle + P_0|\Psi(t)\rangle = c|\Psi'(t)\rangle + P_0|\Psi(t)\rangle,$$

$$c \equiv (||P_1|\Psi(t)\rangle||^2 + ||P_2|\Psi(t)\rangle||^2)^{1/2}.$$

Let the state vector

$$|\Psi'(T)\rangle \equiv U(T-t)|\Psi'(t)\rangle \qquad (1)$$

describe the beam, after passage through the two slits, hitting at the instant $T>t$ a second parallel screen. The map $U \equiv U(T-t)$ is the unitary evolution operator between the two screens.

We picture the second screen similarly as the first: $\mathbf{R}_1 \equiv \{\bar{x}: -\infty < \bar{x} < \infty\}$. There is a detector on this screen

  