Foundations of Physics manuscript No.
(will be inserted by the editor)

Answering Mermin's Challenge

Conservation per No Preferred Reference Frame

W.M. Stuckey · Michael Silberstein · Timothy McDevitt · T.D. Le

Received: date / Accepted: date

Abstract In 1981, Mermin published a now famous paper titled, "Bringing home the atomic world: Quantum mysteries for anybody" that Feynman called, "One of the most beautiful papers in physics that I know." Therein, he presented the "Mermin device" that illustrates the conundrum of entanglement per the Bell spin states for the "general reader." He then challenged the "physicist reader" to explain the way the device works "in terms meaningful to a general reader struggling with the dilemma raised by the device." Herein, we show how the principle of conservation per no preferred reference frame (NPRF) answers that challenge, but still leaves a mystery for those who seek constructive explanation via hidden variables or causal mechanisms. In short, the conservation (SO(3) invariance of the spin measurement outcomes

W.M. Stuckey
Department of Physics
Elizabethtown College
Elizabethtown, PA 17022, USA
E-mail: stuckeym@etown.edu

Michael Silberstein
Department of Philosophy
Elizabethtown College
Elizabethtown, PA 17022, USA
and
Department of Philosophy
University of Maryland
College Park, MD 20742, USA

Timothy McDevitt
Department of Mathematical Science
Elizabethtown College
Elizabethtown, PA 17022, USA

T.D. Le
College of Computing
Georgia Institute of Technology
Atlanta, GA 30332, USA

in the same reference frame) following from the SU(2) symmetry of the Bell spin states holds only *on average* in different reference frames (for different measurements), not on a trial-by-trial basis. Therefore, this "average-only" conservation constitutes an adynamical constraint with no overt evidence for an underlying dynamical mechanism, so we justify it via the principle of NPRF in direct analogy with the postulates of special relativity. Thus, we see a common theme in both relativistic and non-relativistic modern physics relating the fundamental constants c and h, respectively, per principle explanation and the restricted Lorentz symmetry group.

Keywords Mermin device · Bell spin states · principle explanation · entanglement · restricted Lorentz symmetry group

1 Introduction

In 1981, Mermin revealed the conundrum of quantum entanglement for a general audience [1] using his "simple device," which we will refer to as the "Mermin device" (Figure 1). Concerning this paper Feynman wrote to Mermin, "One of the most beautiful papers in physics that I know of is yours in the American Journal of Physics" [2, p. 366-7]. To understand the conundrum of the device required no knowledge of physics, just some simple probability theory, which made the presentation all the more remarkable. In subsequent publications, he "revisited" [3] and "refined" [4] the mystery of quantum entanglement with similarly simple devices. In this paper, we will focus on the original Mermin device as it relates to the mystery of entanglement via the Bell spin states.

The Mermin device functions according to two facts that are seemingly contradictory, thus the mystery. Mermin simply supplies these facts and shows the contradiction, which the "general reader" can easily understand. He then challenges the "physicist reader" to resolve the mystery in an equally accessible fashion for the "general reader." Here is an overview (details are in Section 2).

The "source" in the Mermin device (middle box in Figure 1) creates a pair of particles measured by Alice and Bob in settings 1, 2, or 3 of the boxes on the left and right, respectively, in Figure 1. There are two possible outcomes of a measurement and they are denoted "R" and "G" (indicated by the light bulbs in Figure 1). Whenever Alice and Bob choose to measure their particles using the same setting ("case (a)"), they obtain the same results[1], $\frac{1}{2}$ RR (Alice's outcome is R and Bob's outcome is R) and $\frac{1}{2}$ GG (Alice's outcome is G and Bob's outcome is G). This is Fact 1 about case (a) for his device (Table 1).

Fact 2 says that when making different measurements ("case (b)"), Alice and Bob obtain the same results $\frac{1}{4}$ of the time, $\frac{1}{8}$ RR and $\frac{1}{8}$ GG (Table 1). Mermin then shows how using "instruction sets" to account for Fact 1 violates Fact 2. Concerning the use of instruction sets to account for Fact 1 he writes, "It cannot be proved that there is no other way, but I challenge the reader to suggest any." Essentially, if one uses instruction sets to account for Fact 1, the agreement between Alice and Bob when making different measurements has to be greater than $\frac{1}{3}$ (Bell inequality [6]) in violation of Fact 2, i.e., quantum mechanics violates the Bell inequality. That in a nutshell is the conundrum of entanglement via the Bell spin states as shown by the Mermin device for the "general reader."

Concerning his device Mermin wrote, "Although this device has not been built, there is no reason in principle why it could not be, and probably no insurmountable practical difficulties." Sure enough, the experimental confirmation of the violation of Bell's inequality per quantum entanglement is so

[1] At the end of his paper, Mermin says that Bob's R(G) results actually correspond to Alice's G(R) results per the spin singlet state. We will show that his device maps nicely to the spin triplet states as is, and to the spin singlet state as he suggests. Therefore, his device conveys the mystery of entanglement for any of the Bell spin states. We will focus formally on spin-$\frac{1}{2}$ particles herein, but his device is also valid for spin-1 particles [5].

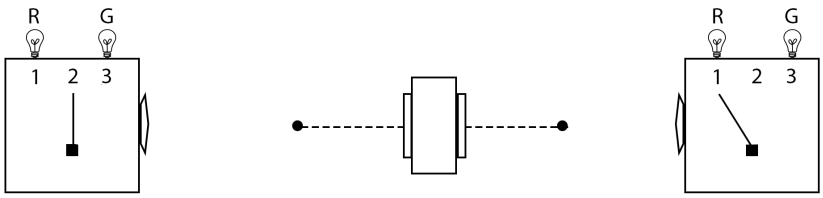

Fig. 1 The Mermin Device. Alice has her measuring device on the left set to 2 and Bob has his measuring device on the right set to 1. The particles have been emitted by the source in the middle and are in route to the measuring devices.

Case (a) Same Settings

		Alice R	Alice G
Bob	R	1/2	0
Bob	G	0	1/2

Case (b) Different Settings

		Alice R	Alice G
Bob	R	1/8	3/8
Bob	G	3/8	1/8

Table 1 Summary of outcome probabilities for the Mermin device.

common that it can now be carried out in the undergraduate physics laboratory [7]. Thus, there is no disputing that the conundrum of the Mermin device has been experimentally well verified, vindicating its prediction by quantum mechanics.

While the conundrum of the Mermin device is now a well-established fact, Mermin's challenge to explain the device "in terms meaningful to a general reader struggling with the dilemma raised by the device" arguably remains unanswered. Of course, what Mermin desires is a true explanation that is compelling and relatively easy to grasp. To answer this challenge, it is generally acknowledged that one needs a compelling model of physical reality or a compelling physical principle by which the conundrum of the Mermin device is resolved. Such a model needs to do more than the "Copenhagen interpretation" [8], which Mermin characterized as "shut up and calculate" [9]. In other words, while the formalism of quantum mechanics accurately predicts the conundrum, the formalism itself does not provide a model of physical reality or underlying physical principle to resolve the conundrum. While there are many interpretations of quantum mechanics, even one published by Mermin [10], there is no consensus among physicists on any given interpretation.

One can speculate as to why there is no consensus view, but one possibility is that, in the language of Einstein, physicists and philosophers are assuming that the most fundamental explanation of quantum entanglement must be "constructive" as opposed to "principle." In what follows Einstein explains the difference between the two [11]:

> We can distinguish various kinds of theories in physics. Most of them are constructive. They attempt to build up a picture of the more complex phenomena out of the materials of a relatively simple formal scheme from which they start out. Thus the kinetic theory of gases seeks to reduce mechanical, thermal, and diffusional processes to movements of molecules – i.e., to build them up out of the hypothesis of molecular motion. When we say that we have succeeded in understanding a group of natural processes, we invariably mean that a constructive theory has been found which covers the processes in question.
>
> Along with this most important class of theories there exists a second, which I will call "principle-theories." These employ the analytic, not the synthetic, method. The elements which form their basis and starting point are not hypothetically constructed but empirically discovered ones, general characteristics of natural processes, principles that give rise to mathematically formulated criteria which the separate processes or the theoretical representations of them have to satisfy. Thus the science of thermodynamics seeks by analytical means to deduce necessary conditions, which separate events have to satisfy, from the universally experienced fact that perpetual motion is impossible.
>
> The advantages of the constructive theory are completeness, adaptability, and clearness, those of the principle theory are logical perfection and security of the foundations. The theory of relativity belongs to the latter class.

In other words, the assumption is that the true "interpretation" of quantum mechanics must be a constructive one in the sense of adverting to fundamental physical entities such as particles or fields and their dynamical equations of motion. Herein we want to consider the possibility that the most fundamental explanation of quantum entanglement is a principle explanation. Our explanation is neutral with respect to constructive ontology in the sense of, for example, particles, fields or waves. However, we are offering a competing account of quantum entanglement for any interpretation that fundamentally explains entanglement in the constructive sense. As Einstein said, this gives us the advantage of "logical perfection and security of the foundations" as our principle account could be true across a number of different constructive interpretations.

Thus, we will share and expand on an underlying physical principle [12] that explains the quantum correlations responsible for the conundrum of the Mermin device. While this explanation, conservation per no preferred reference frame[2] (NPRF), may not be "in terms meaningful to a general reader," it

[2] The term "reference frame" has many meanings in physics related to microscopic and macroscopic phenomena, Galilean versus Lorentz transformations, relatively moving ob-

is pretty close. That is, all one needs to appreciate the explanation is a course in introductory physics, which probably represents the "general reader" interested in this topic.

Of course, entangled states like the Bell spin states result from conservation principles and quantum mechanics produces classical results on average, so this result is perhaps not surprising in a general sense. However, as we will see, the conservation (SO(3) invariance of the Stern-Gerlach (SG) measurement outcomes in the same reference frame) following from the SU(2) symmetry of the Bell spin states holds only *on average* in different reference frames, not on a trial-by-trial basis. Conservation principles are often thought to be underwritten by some dynamical mechanism and hold on a trial-by-trial basis. For example, conservation of momentum obtains when the sum of the forces is zero and Fermat's principle of least time obtains due to refraction per Snell's law, and both hold on a trial-by-trial basis. Thus, the "average-only" SO(3) conservation in different reference frames represented by the SU(2) symmetry of the Bell spin states is a constraint without any obvious deeper dynamical mechanism at work and we will motivate it via the principle of NPRF.

Again, we are answering Mermin's challenge with a principle explanation *a la* special relativity. Indeed, as we will see, the principle of NPRF that we will show leads to the mystery of entanglement per the Bell spin states is also the principle responsible for the mysteries of time dilation and length contraction in special relativity. Since entanglement per the Bell spin states is associated with SO(3) rotations while time dilation and length contraction are associated with Lorentz boosts, we see that NPRF is a "unifying principle" for non-relativistic quantum mechanics and special relativity per the restricted Lorentz symmetry group. In other words, our use of NPRF herein shows how non-relativistic quantum mechanics is already related to special relativity without being Lorentz invariant itself, as with Poincare-invariant quantum field theory.

Despite the fact that our principle explanation reveals an underlying coherence between non-relativistic quantum mechanics and special relativity, some might demand a constructive explanation with its corresponding model of physical reality [13]. Thus, as with special relativity, not everyone will consider our principle account to be explanatory since, "By its very nature such a theory-of-principle explanation will have nothing to say about the reality behind the phenomenon" [14, p. 331]. However, if one accepts special relativity's principle explanation of time dilation and length contraction as fundamental, then they should have no problem accepting conservation per NPRF as a principle answer to Mermin's challenge.

In Section 2, we will provide a detailed account of the Mermin device in terms of spin-$\frac{1}{2}$ measurements[3] (empirical facts) for the "general reader." In Section 3, we will clarify the nature of the SO(3) conservation represented by the SU(2) symmetry of the Bell spin states in terms of the correlation func-

servers, etc. Here, a measurement configuration constitutes a reference frame, as with the light postulate of special relativity.

[3] Unless otherwise specificed, "spin" refers to "spin-$\frac{1}{2}$" herein.

tion (mathematical facts) for the "physicist reader." This allows the "physicist reader" to explain the device "in terms meaningful to a general reader struggling with the dilemma raised by the device" as in Sections 4 & 5. In Section 4, we will explain how the correlation function for the Bell spin states follows from average-only conservation (as a mathematical fact) resulting from the fact that Alice and Bob both always measure $\pm 1\left(\frac{\hbar}{2}\right)$ (quantum), never a fraction of that amount (classical), as shown in Figure 2 (empirical fact). In other words, the mathematical facts (to include average-only conservation) map to the empirical facts (Facts 1 and 2 of the Mermin device), but quantum mechanics does not provide any reason for that mapping, thus the mystery. Therefore, we argue in Section 5 that the SO(3) conservation obtains per NPRF. In Section 5 we show how NPRF is also responsible for the mysteries of time dilation and length contraction in special relativity. Since SO(3) and Lorentz boosts form the restricted Lorentz group, NPRF is a "unifying principle" for non-relativistic quantum mechanics and special relativity per the restricted Lorentz symmetry group (a "precursor" of full unification per Poincare-invariant quantum field theory). This will make it clear how conservation per NPRF provides a principle explanation of the Mermin device for the "general reader."

2 The Mermin Device and Its Conundrum

Here we remind the "general reader" how the Mermin device works and how it relates to the spin measurements carried out with SG magnets and detectors (Figures 2 & 3). The Mermin device contains a source (middle box in Figure 1) that emits a pair of spin-entangled particles towards two detectors (boxes on the left and right in Figure 1) in each trial of the experiment. The settings (1, 2, or 3) on the left and right detectors are controlled by Alice and Bob, respectively, and each measurement at each detector produces either a result of R or G. The following two facts obtain (Table 1):

1. When Alice and Bob's settings are the same in a given trial ("case (a)"), their outcomes are always the same, $\frac{1}{2}$ of the time RR (Alice's outcome is R and Bob's outcome is R) and $\frac{1}{2}$ of the time GG (Alice's outcome is G and Bob's outcome is G).
2. When Alice and Bob's settings are different in a given trial ("case (b)"), the outcomes are the same $\frac{1}{4}$ of the time, $\frac{1}{8}$ RR and $\frac{1}{8}$ GG.

The two possible Mermin device outcomes R and G represent two possible spin measurement outcomes "up" and "down," respectively, (Figure 2) and the three possible Mermin device settings represent three different orientations of the SG magnets (Figures 3 & 4). Mermin writes:

> Why do the detectors always flash the same colors when the switches are in the same positions? Since the two detectors are unconnected there is no way for one to "know" that the switch on the other is set in the same position as its own.

This leads him to introduce "instruction sets" to account for the behavior of the device when the detectors have the same settings. Again, he writes, "It cannot be proved that there is no other way, but I challenge the reader to suggest any." Now look at all trials when Alice's particle has instruction set RRG and Bob's has instruction set RRG, for example.

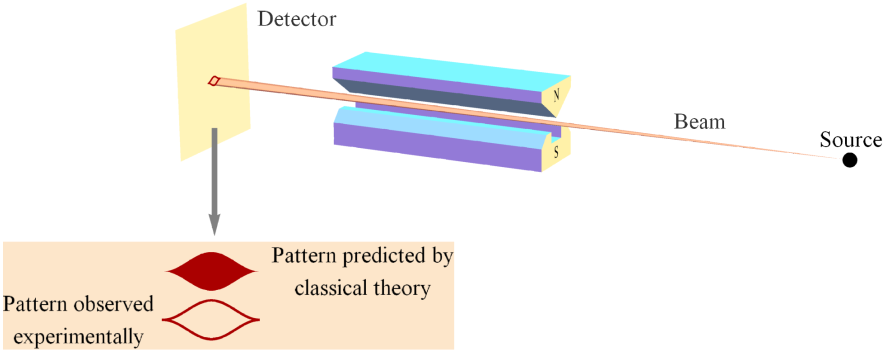

Fig. 2 A Stern-Gerlach (SG) spin measurement showing the two possible outcomes, up and down, represented numerically by $+1$ and -1, respectively. The important point to note here is that the classical analysis predicts all possible deflections, not just the two that are observed. This difference uniquely distinguishes the quantum joint distribution from the classical joint distribution [15].

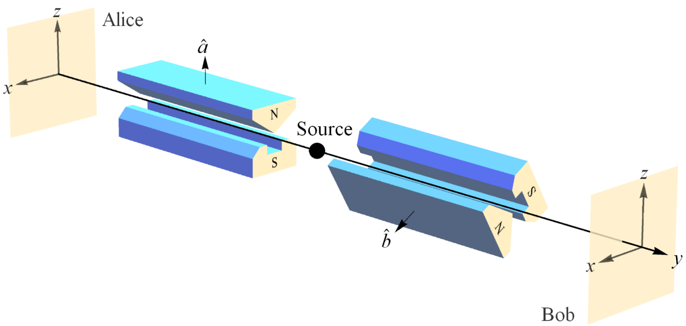

Fig. 3 Alice and Bob making spin measurements on a pair of spin-entangled particles with their Stern-Gerlach (SG) magnets and detectors in the xz-plane.

That means Alice and Bob's outcomes in setting 1 will both be R, in setting 2 they will both be R, and in setting 3 they will both be G. That is,

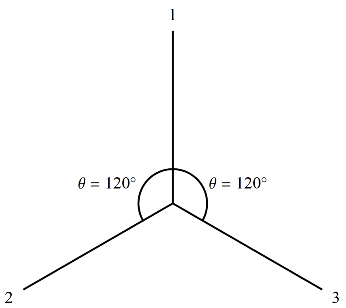

Fig. 4 Three possible orientations of Alice and Bob's SG magnets for the Mermin device.

Case (a) Same Settings

		Alice R	Alice G
Bob	R	1/2	0
Bob	G	0	1/2

Case (b) Different Settings

		Alice R	Alice G
Bob	R	1/4	1/4
Bob	G	1/4	1/4

Table 2 Summary of outcome probabilities for instruction sets. We are assuming the eight possible instruction sets are produced with equal frequency.

the particles will produce an RR result when Alice and Bob both choose setting 1 (referred to as "11"), an RR result when both choose setting 2 (referred to as "22"), and a GG result when both choose setting 3 (referred to as "33"). That is how instruction sets guarantee Fact 1. For different settings Alice and Bob will obtain the same outcomes when Alice chooses setting 1 and Bob chooses setting 2 (referred to as "12"), which gives an RR outcome. And, they will obtain the same outcomes when Alice chooses setting 2 and Bob chooses setting 1 (referred to as "21"), which also gives an RR outcome. That means we have the same outcomes for different settings in 2 of the 6 possible case (b) situations, i.e., in $\frac{1}{3}$ of case (b) trials for this instruction set. This $\frac{1}{3}$ ratio holds for any instruction set with two R(G) and one G(R).

The only other possible instruction sets are RRR or GGG where Alice and Bob's outcomes will agree in $\frac{9}{9}$ of all trials. Thus, the "Bell inequality" for the Mermin device says that instruction sets must produce the same outcomes in more than $\frac{1}{3}$ of all case (b) trials. Indeed, if all eight instruction sets are produced with equal frequency, the RR, GG, RG, and GR outcomes for any given pair of unlike settings (12, 13, 21, 23, 31, or 32) will be produced in equal numbers, so the probability of getting the same outcomes for different settings is $\frac{1}{2}$ (Table 2). But, Fact 2 for quantum mechanics says you only get the same outcomes in $\frac{1}{4}$ of all those trials, thereby violating the prediction per instruction sets. Thus, the conundrum of Mermin's device is that the instruction sets needed for Fact 1 fail to yield the proper outcomes for Fact 2. That quantum mechanics accurately predicts the observed phenomenon without spelling out any means *a la* instruction sets for how it works prompted Smolin to write [16, p. xvii]:

> I hope to convince you that the conceptual problems and raging disagreements that have bedeviled quantum mechanics since its inception are unsolved and unsolvable, for the simple reason that the theory is wrong. It is highly successful, but incomplete.

Of course, this is precisely the complaint leveled by Einstein, Podolsky, and Rosen in their famous paper, "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?" [17]. Our point of course is that quantum mechanics isn't wrong or incomplete, it just needs to be understood as a principle theory, at least as it pertains to the mystery of entanglement.

So, Mermin's challenge to the "physicist reader" is to explain Facts 1 and 2 for the "general reader." We will answer Mermin's challenge by showing that Facts 1 and 2 follow from a very reasonable conservation principle and thereby render Smolin's sentiment entirely misguided. That is, we will see that quantum mechanics is not only complete, but it shares an underlying coherence with Einstein's other revolution [16], special relativity, i.e., the mysteries of both are grounded in the same principle, "no preferred reference frame." The reasonable conservation principle resides in the correlation function, so we start there.

The correlation function between two outcomes over many trials is the average of the two values multiplied together. In this case, there are only two possible outcomes for any setting, +1 (up or R) or −1 (down or G), so the largest average possible is +1 (total correlation, RR or GG, as when the settings are the same) and the smallest average possible is −1 (total anti-correlation, RG or GR). One way to write the equation for the correlation function is

$$\langle \alpha, \beta \rangle = \sum (i \cdot j) \cdot p(i, j \mid \alpha, \beta) \tag{1}$$

where $p(i, j \mid \alpha, \beta)$ is the probability that Alice measures i and Bob measures j given that Alice's SG magnets are at angle α and Bob's SG magnets are at angle β, and $(i \cdot j)$ is just the product of the outcomes i and j. The correlation function for instruction sets for case (a) is the same as that of the Mermin device for case (a), i.e., they're both 1. Thus, we must explore the difference between the correlation function for instruction sets and the Mermin device for case (b).

To get the correlation function for instruction sets for case (b), we need the probabilities of measuring the same outcomes and different outcomes for different settings, so we can use Eq. (1). We saw that when we had two R(G) and one G(R), the probability of getting the same outcomes for different settings was $\frac{1}{3}$ (this would break down to $\frac{1}{6}$ for each of RR and GG overall). Thus, the probability of getting different outcomes would be $\frac{2}{3}$ for these types of instruction sets ($\frac{1}{3}$ for each of RG and GR). That gives a correlation function of

$$\langle \alpha, \beta \rangle = (+1)(+1)\left(\frac{1}{6}\right) + (-1)(-1)\left(\frac{1}{6}\right) + (+1)(-1)\left(\frac{2}{6}\right) + (-1)(+1)\left(\frac{2}{6}\right) = -\frac{1}{3} \quad (2)$$

For the other type of instruction sets, RRR and GGG, we would have a correlation function of $+1$ for different settings, so overall the correlation function for instruction sets for case (b) has to be larger than $-\frac{1}{3}$. Again, if all eight instruction sets are produced with equal frequency, the probability for any particular outcome is $\frac{1}{4}$ for case (b) (Table 2) giving a correlation function of zero. That means the results are uncorrelated as one would expect given that all possible instruction sets are produced randomly. From this we would typically infer that there is nothing that needs to be explained. Indeed, if Fact 1 about case (a) obtains due to some underlying conservation principle, as the symmetries shown in Section 3 suggest, then uncorrelated results for case (b) is more surprising than the anti-correlated results that we now show obtain per the Mermin device. In other words, instruction sets entail there are no observable case (b) consequences for the case (a) conservation. As we now show, the Mermin device says otherwise.

Fact 2 for the Mermin device says the probability of getting the same results (RR or GG) for different settings is $\frac{1}{4}$ ($\frac{1}{8}$ for each of RR and GG, Table 1). Thus, the probability of getting different outcomes for different settings must be $\frac{3}{4}$ ($\frac{3}{8}$ for each of RG and GR, Table 1). That gives a correlation function of

$$\langle \alpha, \beta \rangle = (+1)(+1)\left(\frac{1}{8}\right) + (-1)(-1)\left(\frac{1}{8}\right) + (+1)(-1)\left(\frac{3}{8}\right) + (-1)(+1)\left(\frac{3}{8}\right) = -\frac{1}{2} \quad (3)$$

That means the Mermin device is more strongly anti-correlated for different settings than instruction sets. Indeed, again, if all possible instruction sets are produced with equal frequency, the Mermin device evidences something to explain (anti-correlated results for case (b)) where instruction sets suggest there is nothing in need of explanation (uncorrelated results for case (b)). Again, the Mermin device indicates that the conservation principle responsible for Fact 1 of case (a) has observable implications (Fact 2) for case (b) while instruction sets say we should not expect to see any consequence of Fact 1 for case (b)[4]. Mermin's challenge then amounts to explaining why that is true for the "general reader."

[4] While instruction sets do not predict any observable case (b) consequences for case (a) conservation, other hidden variable accounts do, but none produce the degree of correlation/anti-correlation possessed by quantum mechanics. This is of course Bell's famous result [6].

3 The Bell Spin States

In order to provide that explanation, we first review the nature of conservation at work in the Bell spin states for spin-$\frac{1}{2}$ particles as revealed by the correlation function for the "physicist reader[5]." The Bell spin states are

$$
\begin{aligned}
|\psi_-\rangle &= \frac{|ud\rangle\ - |du\rangle}{\sqrt{2}} \\
|\psi_+\rangle &= \frac{|ud\rangle + |du\rangle}{\sqrt{2}} \\
|\phi_-\rangle &= \frac{|uu\rangle\ - |dd\rangle}{\sqrt{2}} \\
|\phi_+\rangle &= \frac{|uu\rangle + |dd\rangle}{\sqrt{2}}
\end{aligned}
\tag{4}
$$

in the eigenbasis of σ_z. The first state $|\psi_-\rangle$ is called the "spin singlet state" and it represents a total conserved spin angular momentum of zero ($S = 0$) for the two particles involved. The other three states are called the "spin triplet states" and they each represent a total conserved spin angular momentum of one ($S = 1$, in units of $\hbar = 1$). In all four cases, the entanglement represents the conservation of spin angular momentum for the process creating the state. We will be using the Pauli spin matrices to construct our spin measurement operators σ_i. In the eigenbasis of σ_z the Pauli spin matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Finally, all spin matrices have the same eigenvalues of ± 1 and we will denote the corresponding eigenvectors as $|u\rangle$ and $|d\rangle$ for spin up (+1) and spin down (−1), respectively. Using the Pauli spin matrices above with $|u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we see that $\sigma_z|u\rangle = |u\rangle$, $\sigma_z|d\rangle = -|d\rangle$, $\sigma_x|u\rangle = |d\rangle$, $\sigma_x|d\rangle = |u\rangle$, $\sigma_y|u\rangle = i|d\rangle$, and $\sigma_y|d\rangle = -i|u\rangle$. The juxtaposed notation simply means $\sigma_x\sigma_z|ud\rangle = -|dd\rangle$ and $\sigma_x\sigma_y|ud\rangle = -i|du\rangle$, for example. If we flip the orientation of a vector from right pointing (ket) to left pointing (bra) or vice-versa, we transpose and take the complex conjugate. For example, if $|A\rangle = i\begin{pmatrix} 1 \\ 0 \end{pmatrix} = i|u\rangle$, then $\langle A| = -i\begin{pmatrix} 1 & 0 \end{pmatrix} = -i\langle u|$. Thus, any spin matrix can be written as $(+1)|u\rangle\langle u| + (-1)|d\rangle\langle d|$ where $|u\rangle$ and $|d\rangle$ are their up and down eigenvectors, respectively. With that review of the formalism, we proceed to computing the correlation functions for the Bell spin states.

[5] We are providing a comprehensive account here, as not all "physicist readers" are familiar with all the details. The knowledgeable "physicist reader" may skip to the last three paragraphs of this section.

If Alice is making her spin measurement σ_1 in the $\hat{a}$ direction and Bob is making his spin measurement σ_2 in the $\hat{b}$ direction, we have

$$\begin{aligned}\sigma_1 &= \hat{a} \cdot \boldsymbol{\sigma} = a_x\sigma_x + a_y\sigma_y + a_z\sigma_z \\ \sigma_2 &= \hat{b} \cdot \boldsymbol{\sigma} = b_x\sigma_x + b_y\sigma_y + b_z\sigma_z\end{aligned} \tag{5}$$

Using this formalism and the fact that $\{|uu\rangle, |ud\rangle, |du\rangle, |dd\rangle\}$ is an orthonormal set ($\langle uu|uu\rangle = 1$, $\langle uu|ud\rangle = 0$, $\langle du|du\rangle = 1$, etc.), we see that the correlation functions are given by

$$\begin{aligned}\langle\psi_-|\sigma_1\sigma_2|\psi_-\rangle &= -a_xb_x - a_yb_y - a_zb_z \\ \langle\psi_+|\sigma_1\sigma_2|\psi_+\rangle &= a_xb_x + a_yb_y - a_zb_z \\ \langle\phi_-|\sigma_1\sigma_2|\phi_-\rangle &= -a_xb_x + a_yb_y + a_zb_z \\ \langle\phi_+|\sigma_1\sigma_2|\phi_+\rangle &= a_xb_x - a_yb_y + a_zb_z\end{aligned} \tag{6}$$

We now explore the conservation being depicted by the Bell spin states. Let us start with the spin singlet state $|\psi_-\rangle$.

If we transform our basis per

$$\begin{aligned}|u\rangle &\rightarrow \cos(\Theta)|u\rangle + \sin(\Theta)|d\rangle \\ |d\rangle &\rightarrow -\sin(\Theta)|u\rangle + \cos(\Theta)|d\rangle\end{aligned} \tag{7}$$

where Θ is an angle in Hilbert space (as opposed to the SG magnet angles in real space), then $|\psi_-\rangle \rightarrow |\psi_-\rangle$. In other words, $|\psi_-\rangle$ is invariant with respect to this SU(2) transformation. Constructing the corresponding spin measurement operator from these transformed up and down vectors gives

$$|u\rangle\langle u| - |d\rangle\langle d| = \begin{pmatrix}\cos(2\Theta) & \sin(2\Theta) \\ \sin(2\Theta) & -\cos(2\Theta)\end{pmatrix} = \cos(2\Theta)\sigma_z + \sin(2\Theta)\sigma_x \tag{8}$$

So, we see that the invariance of the state under this Hilbert space SU(2) transformation means we have rotational (SO(3)) invariance for the SG measurement outcomes in the xz-plane of real space. Specifically, $|\psi_-\rangle$ says that when the SG magnets are aligned in the z direction (Alice and Bob are in the same reference frame) the outcomes are always opposite ($\frac{1}{2}$ ud and $\frac{1}{2}$ du). Since $|\psi_-\rangle$ has that same functional form under an SU(2) transformation in Hilbert space representing an SO(3) rotation in the xz-plane per Eqs. (7) & (8), the outcomes are always opposite ($\frac{1}{2}$ ud and $\frac{1}{2}$ du) for aligned SG magnets in the xz-plane. That is the SO(3) conservation associated with this SU(2) symmetry. Note that it only deals with case (a) results, i.e., when Alice and Bob are in the same reference frame, so this alone does not distinguish between the Mermin device and instruction sets.

From Eq. (8) we see that when the angle in Hilbert space is Θ, the angle θ of the rotated SG magnets in the xz-plane is $\theta = 2\Theta$. The physical reason for this factor of 2 relating Θ in Hilbert space and θ in real space will be become evident in Section 4 when we reveal the implications of the SO(3) conservation principle for measurements in different reference frames (Figures 6 & 8). Notice that when $\Theta = 45°$, our operator is σ_x, i.e., we have transformed to the eigenbasis of σ_x from the eigenbasis of σ_z.

Another SU(2) transformation that leaves $|\psi_-\rangle$ invariant is

$$\begin{aligned} |u\rangle \rightarrow & \quad \cos(\Theta)|u\rangle + i\sin(\Theta)|d\rangle \\ |d\rangle \rightarrow & \quad i\sin(\Theta)|u\rangle + \cos(\Theta)|d\rangle \end{aligned} \tag{9}$$

Constructing our spin measurement operator from these transformed vectors gives us

$$|u\rangle\langle u| - |d\rangle\langle d| = \begin{pmatrix} \cos(\theta) & -i\sin(\theta) \\ i\sin(\theta) & -\cos(\theta) \end{pmatrix} = \cos(\theta)\sigma_z + \sin(\theta)\sigma_y \tag{10}$$

So, we see that the invariance of the state under this Hilbert space SU(2) transformation means we have rotational (SO(3)) invariance for the SG measurement outcomes in the yz-plane, analogous to what we found for the xz-plane. Notice that when $\Theta = 45°$ our operator is σ_y, i.e., we have transformed to the eigenbasis of σ_y from the eigenbasis of σ_z.

Finally, we see that $|\psi_-\rangle$ is invariant under the third SU(2) transformation

$$\begin{aligned} |u\rangle \rightarrow & \quad (\cos(\Theta) + i\sin(\Theta))|u\rangle \\ |d\rangle \rightarrow & \quad (\cos(\Theta) - i\sin(\Theta))|d\rangle \end{aligned} \tag{11}$$

since this takes $|ud\rangle \rightarrow |ud\rangle$. Constructing our spin measurement operator from these transformed vectors gives us

$$|u\rangle\langle u| - |d\rangle\langle d| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z \tag{12}$$

In other words, Eq. (7) is the Hilbert space SU(2) transformation that represents an SO(3) rotation about the y axis in real space and can be written

$$\begin{pmatrix} u \\ d \end{pmatrix} \rightarrow \begin{pmatrix} \cos(\Theta) & \sin(\Theta) \\ -\sin(\Theta) & \cos(\Theta) \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} = (\cos(\Theta)I + i\sin(\Theta)\sigma_y) \begin{pmatrix} u \\ d \end{pmatrix} \tag{13}$$

Eq. (9) is the Hilbert space SU(2) transformation that represents an SO(3) rotation about the x axis in real space and can be written

$$\begin{pmatrix} u \\ d \end{pmatrix} \rightarrow \begin{pmatrix} \cos(\Theta) & i\sin(\Theta) \\ i\sin(\Theta) & \cos(\Theta) \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} = (\cos(\Theta)I + i\sin(\Theta)\sigma_x) \begin{pmatrix} u \\ d \end{pmatrix} \tag{14}$$

And Eq. (11) is the Hilbert space SU(2) transformation that represents an SO(3) rotation about the z axis in real space and can be written

$$\begin{aligned} \begin{pmatrix} u \\ d \end{pmatrix} \rightarrow \begin{pmatrix} \cos(\Theta) + i\sin(\Theta) & 0 \\ 0 & \cos(\Theta) - i\sin(\Theta) \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} = \\ (\cos(\Theta)I + i\sin(\Theta)\sigma_z) \begin{pmatrix} u \\ d \end{pmatrix} \end{aligned} \tag{15}$$

The SU(2) transformation matrix is often written $e^{i\Theta\sigma_i}$, where $i = \{x, y, z\}$, by expanding the exponential and using $\sigma_i^2 = I$. Since we are in the σ_z eigenbasis, this third transformation means our spin measurement operator is just σ_z. The invariance of $|\psi_-\rangle$ under all three SU(2) transformations makes sense, since the spin singlet state represents the conservation of a total spin angular momentum of $S = 0$, which is directionless, and each SU(2) transformation in Hilbert space corresponds to an element of SO(3) in real space.

So, while we know that invariance under this third SU(2) transformation means we have rotational (SO(3)) invariance of our SG measurement outcomes in the xy-plane, we do not know what those outcomes are unless we rotate our state to one of those eigenbases. That is, we need to know what this state says about the SG measurement outcomes when the SG magnets are aligned in the xy-plane. Since $|\psi_-\rangle$ is invariant under either of the other SU(2) transformations, it has the same form in either the σ_x or σ_y eigenbasis. Thus, the SG measurement outcomes are always opposite ($\frac{1}{2}$ ud and $\frac{1}{2}$ du) for aligned SG magnets in any plane of real space. This will not be the case for the spin triplet state that is invariant under this third SU(2) transformation, as it is *only* invariant under this third SU(2) transformation.

Now, since our state has the same functional form in any plane, we are free to choose any plane we like to compute our correlation function and not lose generality. Let us work in the eigenbasis of $\sigma_1 = \sigma_z$ with $\sigma_2 = \cos(\theta)\sigma_z + \sin(\theta)\sigma_x$ in computing our correlation function for $|\psi_-\rangle$. We have

$$\frac{1}{2}(\langle ud| - \langle du|)\sigma_z[\cos(\theta)\sigma_z + \sin(\theta)\sigma_x](|ud\rangle - |du\rangle) = -\cos(\theta) \tag{16}$$

per the rules of the formalism in agreement with Eq. (6), which gives $-\hat{a} \cdot \hat{b}$. What we see from this analysis is that the conserved spin angular momentum ($S = 0$), being directionless, leads to opposite outcomes for SG magnets at any $\hat{a} = \hat{b}$ and a correlation function of $-\cos(\theta)$ in any plane of real space. As we will see in Section 4, this correlation function tells us there are case (b) implications for our case (a) conservation. Now for the spin triplet states.

We will begin with $|\phi_+\rangle$. The only SU(2) transformation that takes $|\phi_+\rangle \to |\phi_+\rangle$ is Eq. (7). Thus, this state says we have rotational (SO(3)) invariance for our SG measurement outcomes in the xz-plane. Specifically, $|\phi_+\rangle$ says that when the SG magnets are aligned in the z direction (measurements are being made in the same reference frame) the outcomes are always the same ($\frac{1}{2}$ uu and $\frac{1}{2}$ dd). Since $|\psi_+\rangle$ has that same functional form under an SU(2) transformation in Hilbert space representing an SO(3) rotation in the xz-plane per Eqs. (7) & (8), the outcomes are always the same ($\frac{1}{2}$ uu and $\frac{1}{2}$ dd) for aligned SG magnets in the xz-plane. Again, that is the SO(3) conservation associated with this SU(2) symmetry and it applies only to case (a), i.e., measurements made in the same reference frame. In this case, since $|\phi_+\rangle$ is only invariant under Eq. (7), we can only expect rotational invariance for our SG measurement outcomes in the xz-plane. This is confirmed by Eq. (6) where we see that the correlation function for arbitrarily oriented σ_1 and σ_2

is given by $a_x b_x - a_y b_y + a_z b_z$. Thus, unless we restrict our measurements to the xz-plane, we do not have the rotationally invariant correlation function $\hat{a} \cdot \hat{b}$ analogous to the spin singlet state. Restricting our measurements to the xz-plane gives us

$$\frac{1}{2}(\langle uu| + \langle dd|)\sigma_z[\cos(\theta)\sigma_z + \sin(\theta)\sigma_x](|uu\rangle + |dd\rangle) = \cos(\theta) \tag{17}$$

per the rules of the formalism in agreement with Eq. (6). Again, as we will see in Section 4, this correlation function tells us there are case (b) implications for our case (a) conservation. We next consider $|\phi_-\rangle$.

The only SU(2) transformation that leaves $|\phi_-\rangle$ invariant is Eq. (9). Thus, this state says we have rotational (SO(3)) invariance for the SG measurement outcomes in the yz-plane. Since $|\phi_-\rangle$ is only invariant under Eq. (9), we can only expect rotational invariance for our SG measurement outcomes in the yz-plane. This is confirmed by Eq. (6) where we see that the correlation function for arbitrarily oriented σ_1 and σ_2 for $|\phi_-\rangle$ is given by $-a_x b_x + a_y b_y + a_z b_z$. Thus, unless we restrict our measurements to the yz-plane, we do not have the rotationally invariant correlation function $\hat{a} \cdot \hat{b}$ analogous to the spin singlet state. Restricting our measurements to the yz-plane gives us

$$\frac{1}{2}(\langle uu| - \langle dd|)\sigma_z[\cos(\theta)\sigma_z + \sin(\theta)\sigma_y](|uu\rangle - |dd\rangle) = \cos(\theta) \tag{18}$$

per the rules of the formalism in agreement with Eq. (6).

Finally, the only SU(2) transformation that leaves $|\psi_+\rangle$ invariant is Eq. (11). Thus, this state says we have rotational (SO(3)) invariance for our SG measurement outcomes in the xy-plane. But, unlike the situation with $|\psi_-\rangle$, we will need to transform to either the σ_x or σ_y eigenbasis to see what we are going to find in the xy-plane. We can either transform first from the σ_z eigenbasis to the σ_x eigenbasis and then look for our SU(2) invariance transformation, or first transform from the σ_z eigenbasis to the σ_y eigenbasis. We will do both to show how they each work and give self-consistent results.

To go to the σ_x eigenbasis from the σ_z eigenbasis we use Eq. (7) with $\Theta = 45^\circ$

$$|u\rangle \to \frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle \tag{19}$$

$$|d\rangle \to -\frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle \tag{20}$$

This takes $|\psi_+\rangle$ in the σ_z eigenbasis to $-|\phi_-\rangle$ in the σ_x eigenbasis and we know the SU(2) transformation that leaves this invariant is Eq. (9) which then gives a spin measurement operator of $\cos(\theta)\sigma_x + \sin(\theta)\sigma_y$, since we have simply switched the σ_z eigenbasis with the σ_x eigenbasis. Therefore, $|\psi_+\rangle$ says that when the SG magnets are aligned anywhere in the xy-plane the outcomes are always the same ($\frac{1}{2}$ uu and $\frac{1}{2}$ dd). This is consistent with Eq. (6) where we see that the correlation function for arbitrarily oriented σ_1 and σ_2 for $|\psi_+\rangle$ is given by $a_x b_x + a_y b_y - a_z b_z$. Thus, unless we restrict our measurements to the xy-plane, we do not have the rotationally invariant correlation function $\hat{a} \cdot \hat{b}$ analogous to the spin singlet state. Restricting our measurements to the xy-plane gives us

$$\frac{1}{2}(\langle uu| - \langle dd|)\sigma_x[\cos(\theta)\sigma_x + \sin(\theta)\sigma_y](|uu\rangle - |dd\rangle) = \cos(\theta) \tag{21}$$

where $|u\rangle$ and $|d\rangle$ are now the eigenstates for σ_x. That is, $|u\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ and $|d\rangle = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, so that $\sigma_x|u\rangle = |u\rangle$, $\sigma_x|d\rangle = -|d\rangle$, $\sigma_y|u\rangle = i|d\rangle$, and $\sigma_y|d\rangle = -i|u\rangle$. Again, this agrees with Eq. (6). Now let us repeat our analysis of $|\psi_+\rangle$ by first transforming the state into the σ_y eigenbasis.

To go to the σ_y eigenbasis from the σ_z eigenbasis we use Eq. (9) with $\Theta = 45°$

$$|u\rangle \rightarrow \frac{1}{\sqrt{2}}|u\rangle + \frac{i}{\sqrt{2}}|d\rangle \tag{22}$$

$$|d\rangle \rightarrow \frac{i}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle \tag{23}$$

This takes $|\psi_+\rangle$ in the σ_z eigenbasis to $i|\phi_+\rangle$ in the σ_y eigenbasis and we know the transformation that leaves this invariant is Eq. (7) which then gives a spin measurement operator of $\cos(\theta)\sigma_y + \sin(\theta)\sigma_x$, since we have simply switched the σ_z eigenbasis with the σ_y eigenbasis. Again, restricting our SG spin measurements to the xy-plane gives us

$$\frac{-i}{\sqrt{2}}(\langle uu| + \langle dd|)\sigma_y[\cos(\theta)\sigma_y + \sin(\theta)\sigma_x]\frac{i}{\sqrt{2}}(|uu\rangle + |dd\rangle) = \cos(\theta) \tag{24}$$

where $|u\rangle$ and $|d\rangle$ are now the eigenstates for σ_y. That is, $|u\rangle = \begin{pmatrix} -i/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ and $|d\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix}$, so that $\sigma_y|u\rangle = |u\rangle$, $\sigma_y|d\rangle = -|d\rangle$, $\sigma_x|u\rangle = |d\rangle$, and $\sigma_x|d\rangle = |u\rangle$. Again, this agrees with Eq. (6).

What does all this mean? Obviously, the invariance of each of the spin triplet states under its respective SU(2) transformation in Hilbert space represents the conserved spin angular momentum $S = 1$ for each of the planes xz ($|\phi_+\rangle$), yz ($|\phi_-\rangle$), and xy ($|\psi_+\rangle$) in real space. Specifically, when the SG magnets are aligned (the measurements are being made in the same reference frame) anywhere in the respective plane of symmetry the outcomes are always the same ($\frac{1}{2}$ uu and $\frac{1}{2}$ dd). It is a planar conservation according to our analysis and our experiment would determine which plane (see [7] for a spin-1 example). If you want to model a conserved $S = 1$ for some other plane, you simply create a superposition, i.e., expand in the spin triplet basis. And in that plane, you're right back to the mysterious violation of the Bell inequality per conserved spin angular momentum via a correlation function of $\cos(\theta)$, as with any of the spin triplet states, i.e., it is a coordinate-independent fact. In other words, while the SU(2) symmetry marries up with hidden variable accounts when measurements are being made in the same reference frame (both give total correlation or anti-correlation), it entails a stronger correlation/anti-correlation for measurements made in different reference frames than hidden variable accounts. Again, the reason for this, conservation per NPRF, will be explained in Sections 4 & 5.

It is important to note that the conservation at work here deals with the measurement outcomes proper. Per Dakic and Brukner's axiomatic reconstruction of quantum theory [18], the Bell spin states represent measurement outcomes on an entangled pair of "elementary systems." Axiom 1 of their reconstruction states, "An elementary system has the information carrying capacity of at most one bit." Thus, it is not the case that the measurement outcomes are merely the revealed portion of a greater wealth of information carried by an underlying quantum system. Colloquially put, Alice and Bob's measurement outcomes are all that is available.

In conclusion, the correlation function for any pair of case (b) settings in the Mermin device (Figure 4) is $\cos(120°) = -\frac{1}{2}$, in agreement with Eq. (3), instead of zero per that of instruction sets. In other words, the Mermin device represents spin measurements on an $S = 1$ spin-entangled pair of particles in their plane of symmetry at the angles given by Figure 4. If you let Bob's R(G) results represent Alice's G(R) results, the Mermin device then represents spin measurements on an $S = 0$ spin-entangled pair of particles in some plane (all planes are planes of symmetry for $S = 0$). In that case, the correlation function for any pair of case (b) settings in the Mermin device is $-\cos(120°) = \frac{1}{2}$, instead of zero per that of instruction sets. So, for the $S = 0$ case (b) situation, the Mermin device is giving us correlated results rather than uncorrelated results per instruction sets. And, for the $S = 1$ case (b) situation, the Mermin device is giving us anti-correlated results rather than uncorrelated results per instruction sets. We now explain why that is true for the "general reader."

4 The Conservation Principle

Now that we understand the mathematical description of the SO(3) conservation responsible for entanglement per the Bell spin states, it turns out that the "physicist reader" can explain it rather easily to the "general reader." Let us start with the quantum correlation function for the spin singlet state.

Again, the total spin angular momentum is zero and every measurement produces outcomes of $+1$ (up) or -1 (down) in units of $\frac{\hbar}{2} = 1$. Alice and Bob both measure $+1$ and -1 results with equal frequency for any SG magnet angle and when their angles are equal (case (a)) they obtain different outcomes giving total angular momentum of zero. This result is not difficult to understand via conservation of angular momentum, because Alice and Bob's measured values of spin angular momentum cancel directly when $\alpha = \beta$ (Figure 3). But, when

Bob's SG magnets are rotated by $\alpha - \beta = \theta$ relative to Alice's SG magnets (case (b)), we need to clarify the situation.

We have two sets of data, Alice's set and Bob's set. They were collected in N pairs (data events) with Bob's(Alice's) SG magnets at θ relative to Alice's(Bob's). We want to compute the correlation function for these N data events which is

$$\langle \alpha, \beta \rangle = \frac{(+1)_A(-1)_B + (+1)_A(+1)_B + (-1)_A(-1)_B + ...}{N} \tag{25}$$

Now partition the numerator into two equal subsets per Alice's equivalence relation, i.e., Alice's $+1$ results and Alice's -1 results

$$\langle \alpha, \beta \rangle = \frac{(+1)_A(\sum \text{BA+}) + (-1)_A(\sum \text{BA-})}{N} \tag{26}$$

where $\sum$BA+ is the sum of all of Bob's results (event labels) corresponding to Alice's $+1$ result (event label) and $\sum$BA- is the sum of all of Bob's results (event labels) corresponding to Alice's -1 result (event label). Notice this is all independent of the formalism of quantum mechanics. Now, we rewrite that equation as

$$\langle \alpha, \beta \rangle = \frac{(+1)_A(\sum \text{BA+})}{N} + \frac{(-1)_A(\sum \text{BA-})}{N} = \frac{(+1)_A(\sum \text{BA+})}{2\frac{N}{2}} + \frac{(-1)_A(\sum \text{BA-})}{2\frac{N}{2}} \tag{27}$$

which is

$$\langle \alpha, \beta \rangle = \frac{1}{2}(+1)_A\overline{BA+} + \frac{1}{2}(-1)_A\overline{BA-} \tag{28}$$

with the overline denoting average. Again, this correlation function is independent of the formalism of quantum mechanics. All we have assumed is that Alice and Bob measure $+1$ or -1 with equal frequency at any setting in computing this correlation function. Notice that to understand the quantum correlation responsible for Fact 2 of the Mermin device, i.e., the Fact that represents the deviation between the quantum and the classical correlations, we need to understand the origin of $\overline{BA+}$ and $\overline{BA-}$ for the Bell spin states. We now show what that is for the spin singlet state [12], then we extend the argument to the spin triplet states and underwrite it all with NPRF.

In classical physics, one would say the projection of the angular momentum vector of Alice's particle $\mathbf{S}_A = +1\hat{\alpha}$ along $\hat{\beta}$ is $\mathbf{S}_A \cdot \hat{\beta} = +\cos(\theta)$ where again θ is the angle between the unit vectors $\hat{\alpha}$ and $\hat{\beta}$. From Alice's perspective, had Bob measured at the same angle, i.e., $\beta = \alpha$, he would have found the angular momentum vector of his particle was $\mathbf{S}_B = -\mathbf{S}_A = -1\hat{\alpha}$, so that $\mathbf{S}_A + \mathbf{S}_B = \mathbf{S}_{Total} = 0$. Since he did not measure the angular momentum of his particle at the same angle, he should have obtained a fraction of the length of $\mathbf{S}_B$, i.e., $\mathbf{S}_B \cdot \hat{\beta} = -1\hat{\alpha} \cdot \hat{\beta} = -\cos(\theta)$ (Figure 5; this also follows from counterfactual spin measurements on the single-particle state [19]). Of course, Bob only ever obtains +1 or −1, so let us posit that Bob will *average* $-\cos(\theta)$ (Figure 6). This means

$$\overline{BA+} = -\cos(\theta) \tag{29}$$

Likewise, for Alice's $(-1)_A$ results we have

$$\overline{BA-} = \cos(\theta) \tag{30}$$

Putting these into Eq. (28) we obtain

$$\langle \alpha, \beta \rangle = \frac{1}{2}(+1)_A(-\cos(\theta)) + \frac{1}{2}(-1)_A(\cos(\theta)) = -\cos(\theta) \tag{31}$$

which is precisely the correlation function given by quantum mechanics for the spin singlet state (Section 3). Notice that Eqs. (29 & 30) are mathematical facts for obtaining the quantum correlation function, we are simply motivating these facts via conservation of spin angular momentum in accord with the symmetries we discovered in Section 3. Of course, Bob could partition the data according to his equivalence relation (per his reference frame) and claim that it is Alice who must average her results (obtained in her reference frame) to conserve angular momentum. Now for the spin triplet states.

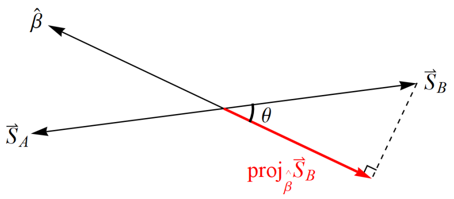

Fig. 5 The angular momentum of Bob's particle $\mathbf{S}_B = -\mathbf{S}_A$ projected along his measurement direction $\hat{\beta}$.

As we saw in Section 3, the spin triplet states represent conservation of spin angular momentum per SO(3) symmetry analogous to the spin singlet state. Thus, we can repeat our story for the $S = 1$ plane of SO(3) rotational invariance, whatever that is. From Alice's perspective, had Bob measured at the same angle, i.e., $\beta = \alpha$, he would have found the angular momentum vector of his particle was $\mathbf{S}_B = \mathbf{S}_A = +1\hat{\alpha}$, so that $\mathbf{S}_A + \mathbf{S}_B = \mathbf{S}_{Total} = 2$ (this is S = 1 in units of $\frac{\hbar}{2} = 1$). Since he did not measure the angular momentum of his particle at the same angle, he should have obtained a fraction of the length of $\mathbf{S}_B$, i.e., $\mathbf{S}_B \cdot \hat{\beta} = +1\hat{\alpha} \cdot \hat{\beta} = \cos(\theta)$ (Figure 7). Of course, Bob only ever obtains +1 or −1, so again let us posit that Bob will *average* $\cos(\theta)$ (Figures 8 & 9). This means

$$\overline{BA+} = \cos(\theta) \tag{32}$$

and similarly

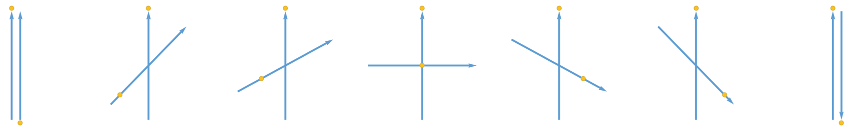

Fig. 6 Average View for the Spin Singlet State. Reading from left to right, as Bob rotates his SG magnets relative to Alice's SG magnets for her +1 outcome, the average value of his outcome varies from −1 (totally down, arrow bottom) to 0 to +1 (totally up, arrow tip). This obtains per conservation of angular momentum on average in accord with no preferred reference frame. Bob can say exactly the same about Alice's outcomes as she rotates her SG magnets relative to his SG magnets for his +1 outcome. That is, their outcomes can only satisfy conservation of angular momentum on average in different reference frames, because they only measure ±1, never a fractional result. Thus, just as with the light postulate of special relativity, we see that no preferred reference frame requires quantum outcomes $\pm 1 \left(\frac{\hbar}{2}\right)$ for all measurements leading to constraint-based explanation, i.e., a fundamentally principle account. Note: Here you can see the physical reason that $\theta = 2\Theta$ for spin-$\frac{1}{2}$ particles found in Section 3, i.e., spin is a bi-directional property in the plane of symmetry for spin-$\frac{1}{2}$ particles.

$$\overline{BA-} = -\cos(\theta) \tag{33}$$

Putting these into Eq. (28) we obtain

$$\langle \alpha, \beta \rangle = \frac{1}{2}(+1)_A(\cos(\theta)) + \frac{1}{2}(-1)_A(-\cos(\theta)) = \cos(\theta) \tag{34}$$

which is the same as the quantum correlation function for the planar $S = 1$ conservation of spin angular momentum that we found in Section 3. Thus, we have an analogous picture for the SO(3) conservation of spin angular momentum for the $S = 1$ states as we had for the $S = 0$ state. Again, we point out that it is simply a mathematical fact that this average conservation principle yields the quantum correlation function. And again, Bob could partition the data according to his equivalence relation (per his reference frame) and claim that it is Alice who must average her results (obtained in her reference frame) to conserve angular momentum.

This all seems rather straightforward, the quantum correlation function for the Mermin device differs from that of instruction sets (classical correlation function) as necessary to satisfy conservation of spin angular momentum on average. And, the reason our conservation principle can only hold on average in different reference frames is because Alice and Bob only measure $\pm 1 \left(\frac{\hbar}{2}\right)$ (quantum), never a fraction of that amount (classical), as shown in Figure 2. Indeed, many physicists are content with this explanation of Facts 1 and 2 for the Mermin device. But, stopping here would ignore what is clearly a conundrum for many others in the foundations community. Therefore, we now articulate for both the "physicist reader" and "general reader" why there is still a mystery and how we propose to resolve it.

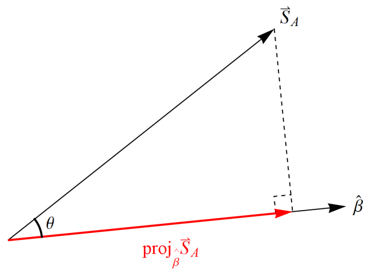

Fig. 7 The angular momentum of Bob's particle $\mathbf{S}_B = \mathbf{S}_A$ projected along his measurement direction $\hat{\beta}$

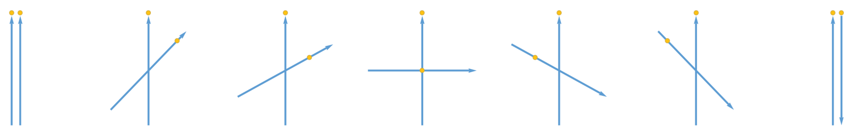

Fig. 8 **Average View for the Spin Triplet States**. Reading from left to right, as Bob rotates his SG magnets relative to Alice's SG magnets for her $+1$ outcome, the average value of his outcome varies from $+1$ (totally up, arrow tip) to 0 to -1 (totally down, arrow bottom). This obtains per conservation of angular momentum on average in accord with no preferred reference frame. Bob can say exactly the same about Alice's outcomes as she rotates her SG magnets relative to his SG magnets for his $+1$ outcome. That is, their outcomes can only satisfy conservation of angular momentum on average in different reference frames, because they only measure ± 1, never a fractional result. Thus, just as with the light postulate of special relativity, we see that no preferred reference frame requires quantum outcomes $\pm 1\left(\frac{\hbar}{2}\right)$ for all measurements leading to constraint-based explanation, i.e., a fundamentally principle account. Note: Here you can see the physical reason that $\theta = 2\Theta$ for spin-$\frac{1}{2}$ particles found in Section 3, i.e., spin is a bi-directional property in the plane of symmetry for spin-$\frac{1}{2}$ particles.

5 Conservation per No Preferred Reference Frame

The problem with the average conservation principle responsible for the quantum correlation function is that it holds *only on average* in different reference frames. Thus, it does not supply an explanation for outcomes on a trial-by-trial basis in different reference frames (Figure 9). This is quite unlike constraints we have in classical physics. For example, conservation of momentum holds on a trial-by-trial basis because the sum of the forces equals zero and a light ray always takes the path of least time (Fermat's principle) because of refraction at the interface per Snell's law. Those constraints hold on average because they hold for each and every trial. In other words, constraints are often explained dynamically and hold on a trial-by-trial basis. Therefore in order to answer Mermin's challenge, we seek something other than a dynamical/causal mechanism to account for this "average-only" conservation in different reference frames. That is, we seek a principle explanation versus a constructive explanation.

We posit that the reason we have average-only conservation in different reference frames is ultimately due to "no preferred reference frame" (NPRF). To motivate NPRF for the Bell spin states, consider the empirical facts. First, Bob and Alice both measure $\pm 1\left(\frac{\hbar}{2}\right)$ for all SG magnet orientations, i.e. in all reference frames. In order to satisfy conservation of spin angular momentum for any given trial when Alice and Bob are making different measurements, i.e., when they are in different reference frames, it would be necessary for Bob or Alice to measure some fraction, $\pm\cos(\theta)$, as we explained in Section 4. For example, if Alice measured $+1$ at $\alpha = 0$ for an $S = 1$ state (in the plane of symmetry) and Bob made his measurement (in the plane of symmetry) at $\beta = 60^\circ$, then Bob's outcome would need to be $\frac{1}{2}$ (Figure 9). In that case, we would know that Alice measured the "true" angular momentum of her particle while Bob only measured a component of the "true" angular momentum for his particle. Thus, Alice's SG magnet orientation would definitely constitute a "preferred reference frame."

But, this is precisely what does *not* happen. Alice and Bob both always measure $\pm 1\left(\frac{\hbar}{2}\right)$, no fractions, in accord with NPRF. And, this fact alone distinguishes the quantum joint distribution from the classical joint distribution [15] (Figure 2). Therefore, the average-only conservation responsible for the correlation function for the Bell spin states leading to Facts 1 and 2 for the Mermin device is actually conservation resulting from NPRF. This is not the only mystery in modern physics resulting from NPRF.

In special relativity, Alice is moving at velocity $\mathbf{V}_a$ relative to a light source and measures the speed of light from that source to be c ($= \frac{1}{\sqrt{\mu_o \epsilon_o}}$, as predicted by Maxwell's equations). Bob is moving at velocity $\mathbf{V}_b$ relative to that same light source and measures the speed of light from that source to be c. Here "reference frame" refers to the relative motion of the observer and source, so all observers who share the same relative velocity with respect to the source occupy the same reference frame. The corresponding transformation here is a Lorentz boost, which with our SO(3) transformation supra form the restricted Lorentz group. NPRF in this context thus means all measurements produce the same outcome c.

As a consequence of this constraint we have time dilation and length contraction, which are then reconciled per NPRF via the relativity of simultaneity. That is, Alice and Bob each partition spacetime per their own equivalence relations (per their own reference frames), so that equivalence classes are their own surfaces of simultaneity. If Alice's equivalence relation over the spacetime events yields the "true" partition of spacetime, then Bob must correct his lengths and times per length contraction and time dilation. Of course, the relativity of simultaneity says that Bob's equivalence relation is as valid as Alice's per NPRF.

This is completely analogous to quantum mechanics, where Alice and Bob each partition the data per their own equivalence relations (per their own reference frames), so that equivalence classes are their own $+1$ and -1 data events. If Alice's equivalence relation over the data events yields the "true" partition of the data, then Bob must correct (average) his results per average-only conservation. Of course, NPRF says that Bob's equivalence relation is as valid as Alice's, which we might call the "relativity of data partition" (Figure 10).

Thus, the mysteries of special relativity (time dilation and length contraction) ultimately follow from the same principle as Mermin's "Quantum mysteries for anybody," i.e., no preferred reference frame. So, if one accepts special relativity's principle explanation of time dilation and length contraction, then they should have no problem accepting conservation per NPRF as a principle answer to Mermin's challenge. Loosely speaking, NPRF is a "unifying principle" for non-relativistic quantum mechanics and special relativity per the restricted Lorentz symmetry group.

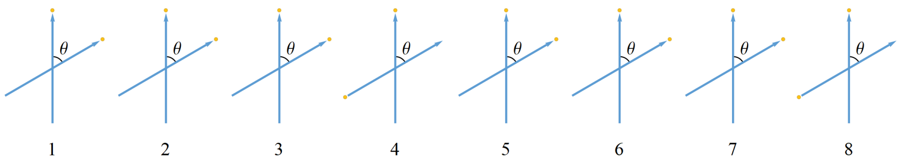

Fig. 9 A spatiotemporal ensemble of 8 experimental trials for the spin triplet states showing Bob's outcomes corresponding to Alice's $+1$ outcomes when $\theta = 60°$. Angular momentum is not conserved in any given trial, because there are two different measurements being made, i.e., outcomes are in two different reference frames, but it is conserved on average for all 8 trials (six up outcomes and two down outcomes average to $\cos 60° = \frac{1}{2}$). It is impossible for angular momentum to be conserved explicitly in each trial since the measurement outcomes are binary (quantum) with values of $+1$ (up) or -1 (down) per no preferred reference frame. The SO(3) conservation principle at work here does not assume Alice and Bob's measured values of angular momentum are mere components of some hidden angular momentum (Figures 5 & 7). That is, the measured values of angular momentum *are* the angular momenta contributing to this SO(3) conservation, as we saw in Section 3.

6 Discussion

We have shown how the "physicist reader" might explain the Mermin device "in terms meaningful to a general reader struggling with the dilemma raised by the device" per Mermin's challenge. Essentially, we argued that the "average-only" conservation of spin angular momentum between different reference frames for the Bell spin states obtains due to "no preferred reference frame" (NPRF). Thus, the mystery of entanglement per the Bell spin states results from the same principle (NPRF) as the mysteries of time dilation and length contraction per special relativity. Indeed, one can say the Lorentz transformations and the Bell spin states can be *derived* from no preferred reference frame, since NPRF gives the postulates of special relativity whence the Lorentz transformations and NPRF gives the quantum correlation functions whence the spin-$\frac{1}{2}$ and spin-1 singlet and triplet states [5]. So, if one accepts special

Special Relativity	Quantum Mechanics
Empirical Fact: Alice and Bob both measure *c*, regardless of their relative motion	Empirical Fact: Alice and Bob both measure +1/-1 $\left(\frac{\hbar}{2}\right)$, regardless of their relative SG orientation
Alice(Bob) says of Bob(Alice): Time dilation and length contraction	Alice(Bob) says of Bob(Alice): Must average results
NPRF: Relativity of simultaneity	NPRF: Relativity of data partition

Fig. 10 Comparing special relativity with quantum mechanics according to no preferred reference frame (NPRF). Because Alice and Bob both measure the same speed of light c regardless of their relative motion per NPRF, Alice(Bob) may claim that Bob's(Alice's) length and time measurements are erroneous and need to be corrected (length contraction and time dilation). Likewise, because Alice and Bob both measure the same values for spin angular momentum $\pm 1 \left(\frac{\hbar}{2}\right)$ regardless of their relative SG magnet orientation per NPRF, Alice(Bob) may claim that Bob's(Alice's) individual ± 1 values are erroneous and need to be corrected (averaged, Figures 6, 8, & 9). In both cases, NPRF resolves the mystery it creates. In special relativity, the apparently inconsistent results can be reconciled via the relativity of simultaneity. That is, Alice and Bob each partition spacetime per their own equivalence relations (per their own reference frames), so that equivalence classes are their own surfaces of simultaneity and these partitions are equally valid per NPRF. This is completely analogous to quantum mechanics, where the apparently inconsistent results per the Bell spin states arising because of NPRF can be reconciled by NPRF via the "relativity of data partition." That is, Alice and Bob each partition the data per their own equivalence relations (per their own reference frames), so that equivalence classes are their own $+1$ and -1 data events and these partitions are equally valid.

relativity's principle explanation of time dilation and length contraction, then they should have no problem accepting conservation per NPRF as a principle answer to Mermin's challenge. Loosely speaking, NPRF is a "unifying principle" for non-relativistic quantum mechanics and special relativity per the restricted Lorentz symmetry group (a "precursor" of full unification per Poincare-invariant quantum field theory).

Of course, not everyone will require such a unifying principle. For example, one might simply rely on SO(3) rotations as an explanation of Bell spin state entanglement, and Lorentz boosts as an explanation of length contraction and time dilation, without invoking NPRF to relate the two transformations. Indeed, even Mermin explained length contraction using Lorentz boosts and electromagnetic forces without further justification before writing [20, p. 228]:

> [G]iven the natural laws and transformation rules of any one observer, we can deduce that any other observer will describe his version of the facts by the same laws and same transformation rules.
>
> This is the structure of physical law as far as it is understood today. I hope some hint of its engrossing and majestic beauty has been suggested here. For it is ultimately just this overwhelming beauty that renders the final, unanswerable "why" superfluous.

And we would agree that NPRF is superfluous in the context of special relativity alone. That is, given the postulates of special relativity alone, NPRF is redundant. But, we are not offering NPRF as an explanation of the postulates of special relativity alone. We are using NPRF to relate non-relativistic quantum mechanics and special relativity via their famous mysteries, making the value judgement that increased unification and coherence between these theories reveals yet more "overwhelming beauty" in physics. The point is, we are hypothesizing that the SO(3) symmetry with average-only conservation as an explanation of Bell state entanglement, and Lorentz symmetry with relativity of simultaneity as an explanation of length contraction and time dilation, are expressions of a deeper truth, NPRF, with seemingly disparate multiple physical consequences. We suggest perhaps other unresolved phenomena in physics might be explained in a similar fashion [21].

There was resistance to Einstein's light postulate because he posited something as axiomatic that most people wanted to have explained constructively. For example, Lorentz complained [22, p. 230]:

> Einstein simply postulates what we have deduced, with some difficulty and not altogether satisfactorily, from the fundamental equations of the electromagnetic field.

And Michelson said [23]:

> It must be admitted, these experiments are not sufficient to justify the hypothesis of an ether. But then, how can the negative result be explained?

In other words, even Lorentz and Michelson were apparently not satisfied with special relativity's principle explanation, as they required some 'deeper mechanism' to explain why observers measure the same speed of light c in different reference frames.

Likewise here, if one requires some 'deeper mechanism' to explain conservation per NPRF, then this constraint is simply one mystery replacing another. Therefore, we understand not everyone will agree that our principle explanation, conservation per NPRF, answers Mermin's challenge even though it reveals an underlying coherence between non-relativistic quantum mechanics and special relativity. Perhaps Lorentz would agree at this point, since he did acknowledge [22, p. 230]:

> By doing so, [Einstein] may certainly take credit for making us see in the negative result of experiments like those of Michelson, Rayleigh, and Brace, not a fortuitous compensation of opposing effects but the manifestation of a general and fundamental principle.

We should note that we are not alone in making such a suggestion, there is also interest in finding such fundamental principles for quantum mechanics in the quantum information community [5]. For example, Fuchs writes [24, p. 285]:

> Compare [quantum mechanics] to one of our other great physical theories, special relativity. One could make the statement of it in terms of some very crisp and clear physical principles: The speed of light is constant in all inertial frames, and the laws of physics are the same in all inertial frames. And it struck me that if we couldn't take the structure of quantum theory and change it from this very overt mathematical speak – something that didn't look to have much physical content at all, in a way that anyone could identify with some kind of physical principle – if we couldn't turn that into something like this, then the debate would go on forever and ever. And it seemed like a worthwhile exercise to try to reduce the mathematical structure of quantum mechanics to some crisp physical statements.

So, quantum information theorists seek "the *reconstruction* of quantum theory" via a constraint-based/principle approach [25].

The bottom line is that a compelling constraint (who would argue with conservation per NPRF?) answers Mermin's challenge without any obvious corresponding 'dynamical/causal influence' or hidden variables to account for the results on a trial-by-trial basis. By accepting this principle explanation as fundamental, the lack of a compelling, consensus constructive explanation is not a problem. This is just one of many mysteries in physics created by dynamical thinking and resolved by constraint-based thinking [21]. We understand that the dynamical and causal nature of our everyday experience creates a bias in favor of constructive explanation, but perhaps it is time to question that bias as a matter of principle.

Acknowledgements We thank honors students in introductory physics at Elizabethtown College for serving as "general readers" of the manuscript.

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. N. Mermin, American Journal of Physics **49**(10), 940 (1981)
2. M. Feynman, *Perfectly Reasonable Deviations from the Beaten Track: The Letters of Richard P. Feynman* (Basic Books, New York, 2005)
3. N. Mermin, American Journal of Physics **58**(8), 731 (1990)
4. N. Mermin, American Journal of Physics **62**(10), 880 (1994)
5. W. Stuckey, M. Silberstein, T. McDevitt, I. Kohler, Entropy **21**, 692 (2019). https://arxiv.org/abs/1807.09115
6. J. Bell, Physics **1**, 195 (1964)
7. D. Dehlinger, M. Mitchell, American Journal of Physics **70**(9), 903 (2002)
8. A. Becker, *What is Real? The Unfinished Quest for the Meaning of Quantum Physics* (Basic Books, New York, 2018)
9. N. Mermin, Physics Today **57**(5), 10 (2004)
10. N. Mermin, American Journal of Physics **66**(9), 753 (1998)

11. A. Einstein, London Times pp. 53–54 (1919)
12. C. Unnikrishnan, Europhysics Letters **69**, 489 (2005). https://arxiv.org/abs/quant-ph/0407041
13. L. Felline, Philosophy of Science **78**, 989 (2011)
14. Y. Balashov, M. Janssen, British Journal for the Philosophy of Science **54**, 327 (2003)
15. A. Garg, N. Mermin, Phys. Rev. Lett. **49**, 901 (1982)
16. L. Smolin, *Einstein's Unfinished Revolution: The Search for What Lies Beyond the Quantum* (Penguin Press, New York, 2019)
17. A. Einstein, B. Podolsky, N. Rosen, Phys. Rev. **47**, 777 (1935). DOI 10.1103/PhysRev.47.777. https://link.aps.org/doi/10.1103/PhysRev.47.777
18. B. Dakic, C. Brukner. Quantum theory and beyond: Is entanglement special? (2009). https://arxiv.org/abs/0911.0695
19. S. Boughn. Making sense of Bell's theorem and quantum nonlocality (2017). https://arxiv.org/abs/1703.11003
20. N. Mermin, *Space and Time in Special Relativity* (Waveland Press, Inc., Prospect Heights, IL, 1968)
21. M. Silberstein, W. Stuckey, T. McDevitt, *Beyond the Dynamical Universe: Unifying Block Universe Physics and Time as Experienced* (Oxford University Press, Oxford, UK, 2018)
22. H. Lorentz, *The Theory of Electrons and Its Applications to the Phenomena of Light and Radiant Heat* (G.E. Stechert and Co., New York, USA, 1916)
23. A. Michelson. The mechanical universe episode 41: The michelson-morley experiment (1985)
24. C. Fuchs, B. Stacey, in *Quantum Theory: Informational Foundations and Foils*, ed. by G. Chiribella, R. Spekkens (Springer, Dordrecht, 2016), pp. 283–305
25. G. Chiribella, R. Spekkens, in *Quantum Theory: Informational Foundations and Foils*, ed. by G. Chiribella, R. Spekkens (Springer, Dordrecht, 2016), pp. 1–18