Schrödinger fm
Klein-Gordon eqn.

Chapter III.5

Field Theory without Relativity

Slower in its maturity

Quantum field theory at its birth was relativistic. Later in its maturity, it found applications in condensed matter physics. We will have a lot more to say about the role of quantum field theory in condensed matter, but for now, we have the more modest goal of learning how to take the nonrelativistic limit of a quantum field theory.

The Lorentz invariant scalar field theory

$$\mathcal{L} = (\partial\Phi^\dagger)(\partial\Phi) - m^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2 \tag{1}$$

(with $\lambda > 0$ as always) describes a bunch of interacting bosons. It should certainly contain the physics of slowly moving bosons. For clarity consider first the relativistic Klein-Gordon equation

$$(\partial^2 + m^2)\Phi = 0 \tag{2}$$

for a free scalar field. A mode with energy $E = m + \varepsilon$ would oscillate in time as $\Phi \propto e^{-iEt}$. In the nonrelativistic limit, the kinetic energy ε is much smaller than the rest mass m. It makes sense to write $\Phi(\vec{x}, t) = e^{-imt}\varphi(\vec{x}, t)$, with the field φ oscillating much more slowly than e^{-imt} in time. Plugging into (2) and using the identity $(\partial/\partial t)e^{-imt}(\cdots) = e^{-imt}(-im + \partial/\partial t)(\cdots)$ twice, we obtain $(-im + \partial/\partial t)^2\varphi - \vec{\nabla}^2\varphi + m^2\varphi = 0$. Dropping the term $(\partial^2/\partial t^2)\varphi$ as small compared to $-2im(\partial/\partial t)\varphi$, we find Schrödinger's equation, as we had better:

$$i\frac{\partial}{\partial t}\varphi = -\frac{\vec{\nabla}^2}{2m}\varphi \tag{3}$$

By the way, the Klein-Gordon equation was actually discovered before Schrödinger's equation.

Having absorbed this, you can now easily take the nonrelativistic limit of a quantum field theory. Simply plug

$$\Phi(\vec{x}, t) = \frac{1}{\sqrt{2m}}e^{-imt}\varphi(\vec{x}, t) \tag{4}$$