ROBOTICS REVIEW

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A.I. DUPONT RESEARCH INSTITUTE
APPLIED SCIENCE AND ENGINEERING GROUP

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AGENDA

MATHMATICAL PRELIMINARY

FORWARD KINEMATICS

INVERSE KINEMATICS

DIFFERENTIAL MOTION (VELOCITY)

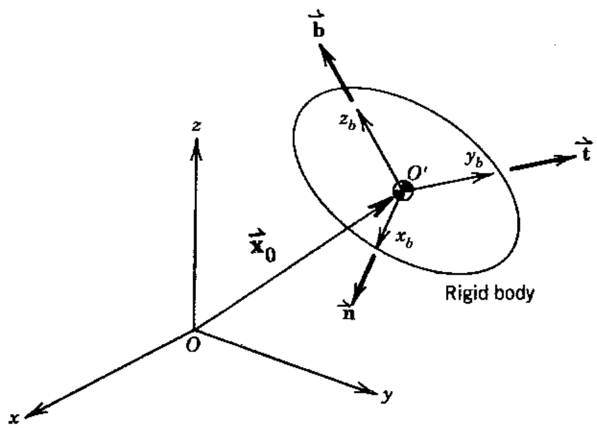
STATICS (FORCES AND COMPLIANCE)

DYNAMICS (CONCEPTUAL OVERVIEW)

SYSTEMS MODELING AND CONTROL
(CONCEPTUAL OVERVIEW)

Mathematical Preliminary

The arm linkage of a manipulator can be modeled as a system of rigid bodies. The location of each single rigid body is completely described by its *position* and *orientation*.

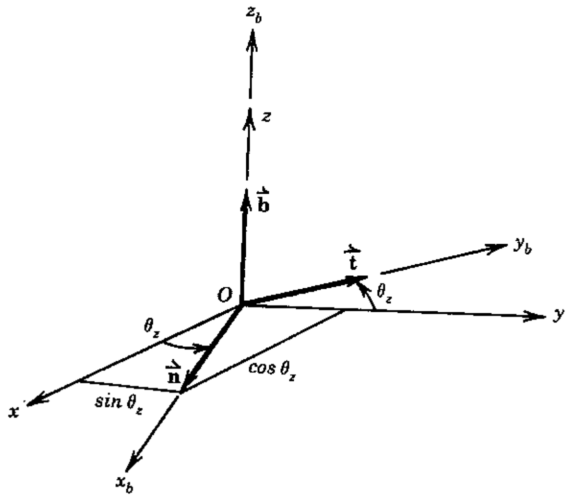


Position and orientation of a rigid body.

Let $\vec{\mathbf{n}}$, $\vec{\mathbf{t}}$ and $\vec{\mathbf{b}}$ be unit vectors pointing the directions of the coordinate axes, x_b , y_b and z_b, respectively. The components of each unit vector are direction cosines of each coordinate axis projected into the fixed coordinate frame O–xyz.

$$\vec{\mathbf{x}} = \vec{\mathbf{x}}_0 + \mathbf{R}\vec{\mathbf{x}}^b \qquad \mathbf{R} = [\, \vec{\mathbf{n}}, \vec{\mathbf{t}}, \vec{\mathbf{b}} \,]$$

provides the desired coordinate transformation from the body coordinates $\vec{\mathbf{x}}^b$ to the fixed coordinates $\vec{\mathbf{x}}$.



Example

As shown in Figure 2-3, the origin of coordinate frame $O'-x_b y_b z_b$ coincides with the origin of the fixed frame $O-xyz$. The angle between axes x and x_b is denoted by $\theta_z = \angle xOx_b$. Axis z_b, on the other hand, coincides with axis z. Let us find the vector $\vec{\mathbf{x}}_0$ and the matrix $\mathbf{R}$ that represent the position and orientation of frame $O'-x_b y_b z_b$ relative to frame $O-xyz$, and then obtain the coordinate transformation from $O'-x_b y_b z_b$ to $O-xyz$.

Since the origins of the two coordinate frames coincide, position vector $\vec{\mathbf{x}}_0$ is zero. To obtain the rotation matrix $\mathbf{R}$, let us find the three unit vectors, $\vec{\mathbf{n}}$, $\vec{\mathbf{t}}$, and $\vec{\mathbf{b}}$, composing $\mathbf{R}$. As shown in Figure 2-3, the components of each vector are its direction cosines with respect to $O-xyz$. Therefore,

$$\vec{\mathbf{n}} = \begin{pmatrix} \cos\theta_z \\ \sin\theta_z \\ 0 \end{pmatrix} \qquad \vec{\mathbf{t}} = \begin{pmatrix} -\sin\theta_z \\ \cos\theta_z \\ 0 \end{pmatrix} \qquad \vec{\mathbf{b}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

so that

$$\mathbf{R} = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The coordinate transformation is then obtained by substituting the matrix $\mathbf{R}$ and $\mathbf{x}_0 = 0$ into equation (2-9). The components of the transformation expressions are thus given by

$\vec{X} = \vec{X}_0 + R\vec{X}^{body}$ $\qquad$ $\vec{X}^b = [u, v, w]$

$$x = u\cos\theta_z - v\sin\theta_z$$

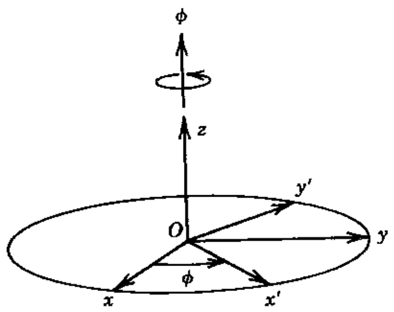
$$y = u\sin\theta_z + v\cos\theta_z$$

$$z = w$$

Euler Angles

(independent, in that each of them can vary arbitrarily)

The three consecutive rotations used to define the Euler angles:



(a)

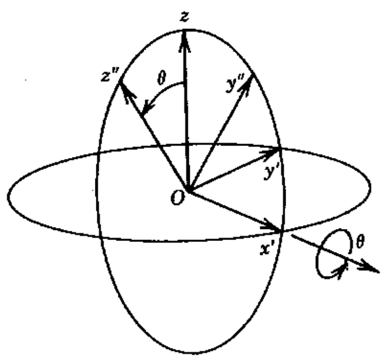
Coordinates $\vec{\mathbf{x}}' = [x', y', z]^T$ are transformed to coordinates $\vec{\mathbf{x}} = [x, y, z]^T$ by the 3 × 3 rotation matrix $\mathbf{R}_z(\phi)$, which is defined as

$$\vec{\mathbf{x}} = \mathbf{R}_z(\phi)\vec{\mathbf{x}}'$$

$$\mathbf{R}_z(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly, the coordinate transformation from $\mathbf{x}'' = [x', y'', z'']^T$ to $\mathbf{x}'$ associated with rotation θ is given by

$$\vec{\mathbf{x}}' = \mathbf{R}_{x'}(\theta)\vec{\mathbf{x}}''$$



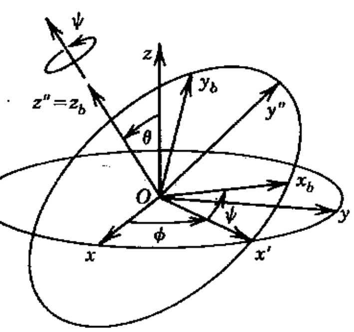
(b)

where

$$\mathbf{R}_{x'}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Finally, for the rotation ψ, we have

$$\vec{\mathbf{x}}'' = \mathbf{R}_{z''}(\psi)\vec{\mathbf{x}}^b$$



(c)

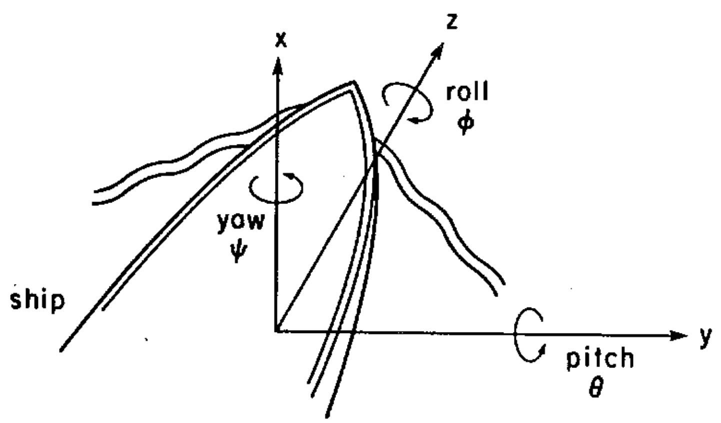
$$\mathbf{R}_{z''}(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

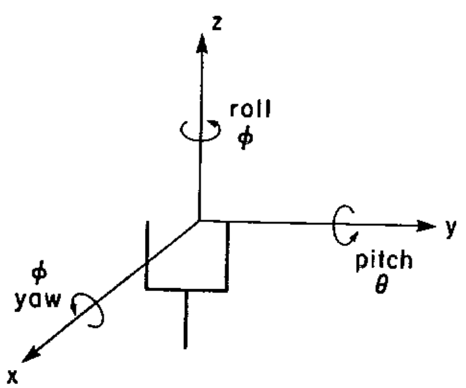
Combining the three coordinate transformations yields

$$\vec{\mathbf{x}} = \mathbf{R}_z(\phi)\,\mathbf{R}_{x'}(\theta)\,\mathbf{R}_{z''}(\psi)\,\vec{\mathbf{x}}^b$$

Roll, Pitch, and Yaw

Orientation is more frequently specified by a sequence of rotations about the x, y, or z axes. Euler angles describe any possible orientation in terms of a rotation ϕ about the z axis, then a rotation θ about the new y axis, y', and finally, a rotation about the new z axis, z'', of ψ.





recall the coordinate transformation given by

$$\vec{\mathbf{x}} = \vec{\mathbf{x}}_0 + \mathbf{R}\vec{\mathbf{x}}^b$$

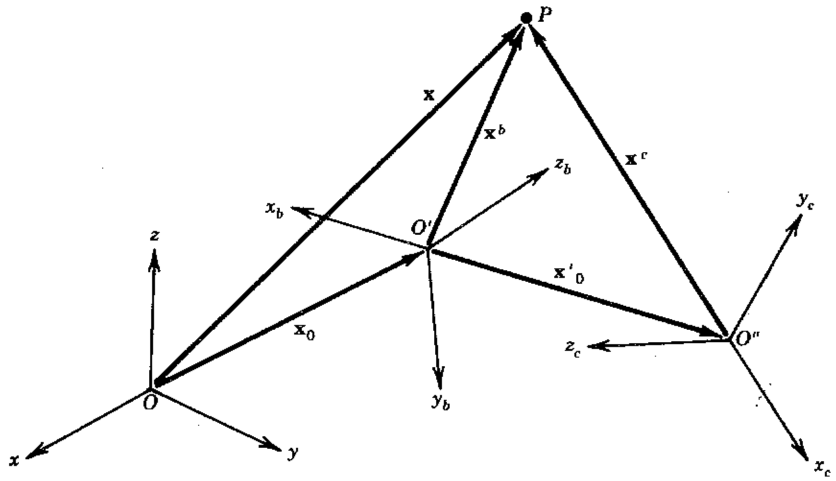
Equation can be written as

$$\vec{\mathbf{X}} = \mathbf{A}\,\vec{\mathbf{X}}^b$$

that is,

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{R} & \vec{\mathbf{x}}_0 \\ \hline \mathbf{O} & 1 \end{array}\right] \begin{bmatrix} u \\ v \\ w \\ 1 \end{bmatrix}$$

The compactness of the homogeneous transformation is particularly advantageous when we represent consecutive transformations:

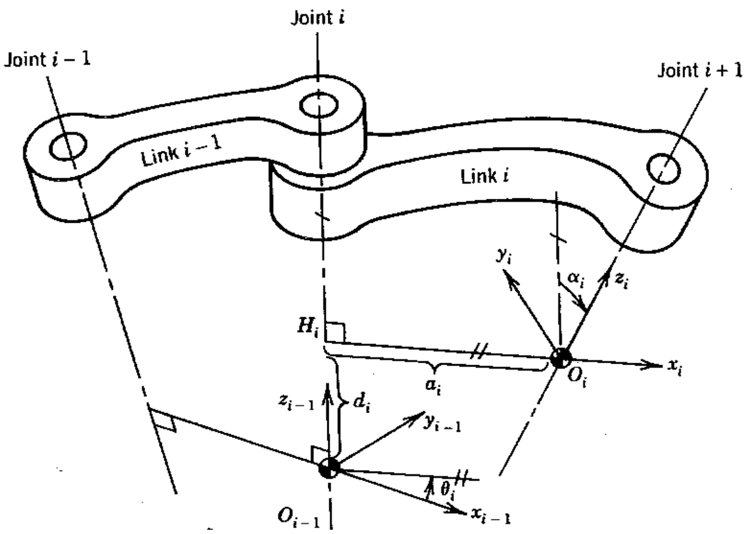


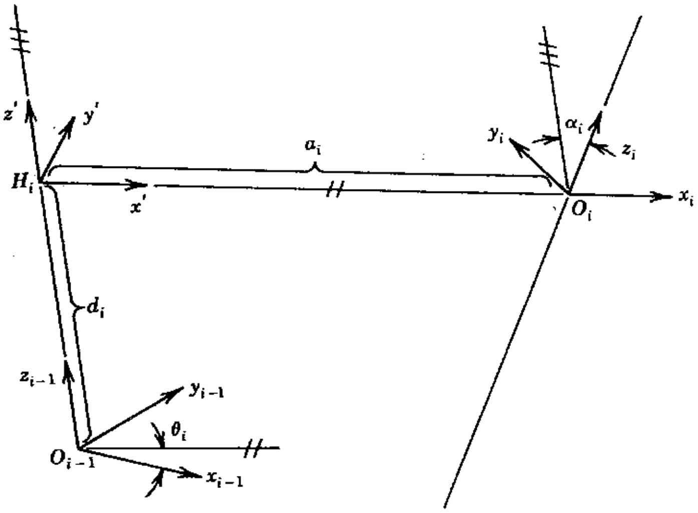
$$\vec{\mathbf{x}} = \vec{\mathbf{x}}_0 + \mathbf{R}\vec{\mathbf{x}}'_0 + \mathbf{R}\mathbf{R}'\vec{\mathbf{x}}^c \qquad\qquad \vec{\mathbf{x}}^b = \vec{\mathbf{x}}'_0 + \mathbf{R}'\vec{\mathbf{x}}^c$$

$$\vec{\mathbf{X}} = \mathbf{A}_1^0\mathbf{A}_2^1 \cdots \mathbf{A}_n^{n-1}\vec{\mathbf{X}}^n$$

The Denavit-Hartenberg Notation

(minimum number of parameters to completely describe the kinematic relationship)





the coordinate transformation from $\mathbf{X}^i$ to $\mathbf{X}'$ is given by

$$\mathbf{X}' = \mathbf{A}_i^{int}\mathbf{X}^i$$

where

$$\mathbf{A}_i^{int} = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Similarly the transformation from $\mathbf{X}'$ to $\mathbf{X}^{i-1}$ is given by

$$\mathbf{X}^{i-1} = A_{int}^{i-1}\,\mathbf{X}'$$

where

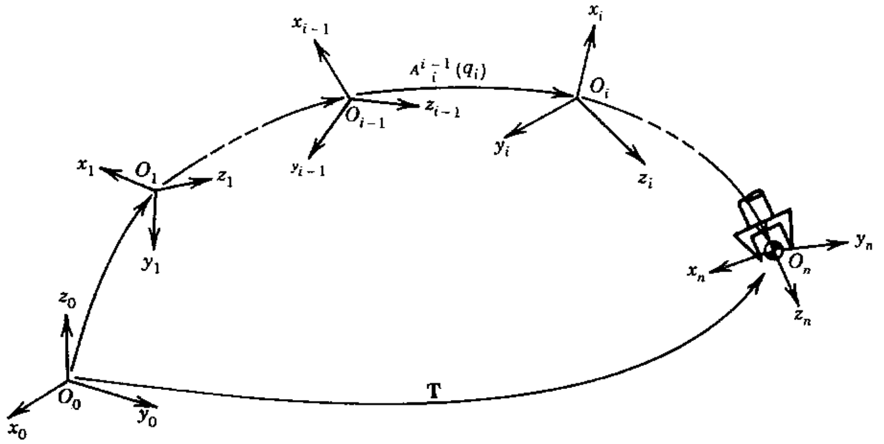
$$\mathbf{A}_{i-1}^{int} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Combining equations leads to

$$\mathbf{X}^{i-1} = \mathbf{A}_i^{i-1}\,\mathbf{X}^i$$

where

$$\mathbf{A}_i^{i-1} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



The representation of the end-effector location by a 4x4 matrix.

orientation of the last link relative to the base frame is given by

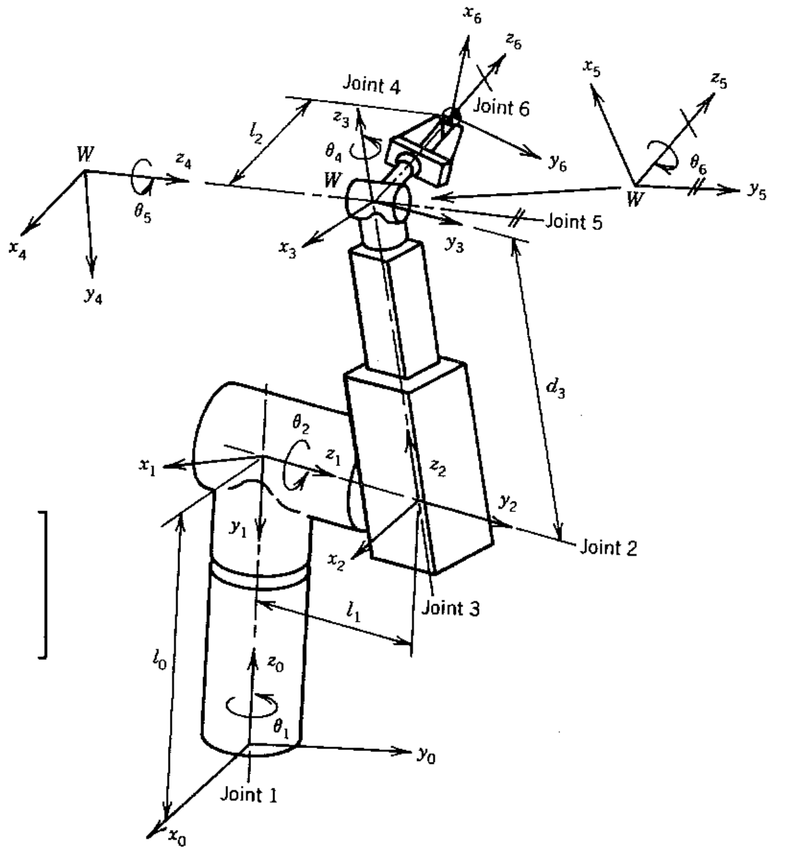
$$\mathbf{T} = \mathbf{A}_1^0(q_1)\mathbf{A}_2^1(q_2) \cdots \mathbf{A}_n^{n-1}(q_n) \qquad \begin{array}{ll} q_i = \theta_i & \text{for a revolute joint} \\ q_i = d_i & \text{for a prismatic joint} \end{array}$$

where $\mathbf{T}$ is a 4×4 matrix representing the position and orientation of the last link with reference to the base frame, as shown in Figure 2-11. Equation (2-36) provides the functional relationship between the last link position and orientation and the displacements of all the joints involved in the open kinematic chain. It is referred to as the *kinematic equation* of the manipulator arm, and governs the fundamental kinematic behavior of the arm.

Denavit-Hartenberg Notation

Example

The Kinematic Model of a 5-R-1-P Manipulator Arm.



$$\mathbf{T} = \mathbf{A}_1^0(\theta_1)\mathbf{A}_2^1(\theta_2)\mathbf{A}_3^2(d_3)\mathbf{A}_4^3(\theta_4)\mathbf{A}_5^4(\theta_5)\mathbf{A}_6^5(\theta_6)$$

$$\mathbf{A}_i^{i-1} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}_1^0(\theta_1) = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & l_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Link parameters for the 5-R-1-P manipulator.

link number	α_i	a_i	d_i	θ_i
1	-90°	0	l_0	θ_1
2	+90°	0	l_1	θ_2
3	0	0	d_3	0
4	-90°	0	0	θ_4
5	+90°	0	0	θ_5
6	0	0	l_2	θ_6

$$d_3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}_5^4(\theta_5) = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

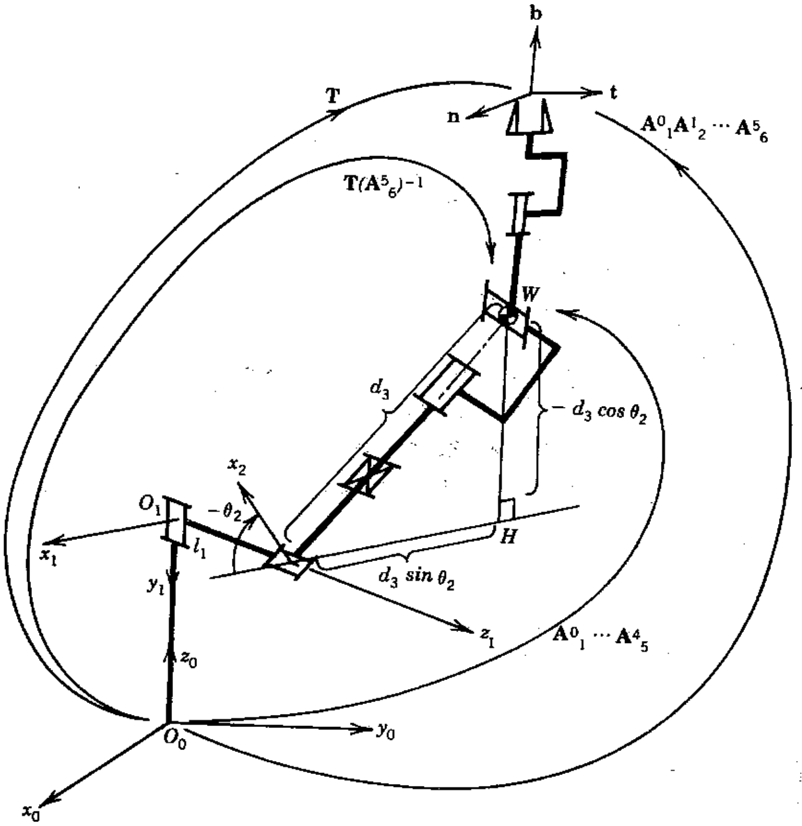
$$\mathbf{A}_2^1(\theta_2) = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}_4^3(\theta_4) = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}_6^5(\theta_6) = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & l_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $c_i = \cos(\theta_i)$ and $s_i = \sin(\theta_i)$

Inverse Kinematics



$$T = A_1^0 A_2^1 A_3^2 A_4^3 A_5^4 A_6^5$$

$$T(A_6^5)^{-1} = A_1^0 A_2^1 A_3^2 A_4^3 A_5^4$$

Further, a more convenient form of simultaneous equations can be derived by evaluating the fourth columns of

$$(A_1^0)^{-1} T (A_6^5)^{-1} = A_2^1 A_3^2 A_4^3 A_5^4$$

The fourth column vector of the right-hand side represents the position of W with respect to the first coordinate frame through the arm linkage, as shown in Figure [redacted], and is simply given by

$$x_W^1 = \begin{pmatrix} d_3 s_2 \\ -d_3 c_2 \\ l_1 \end{pmatrix}$$

Thus, writing the desired end-effector position and orientation $T = \begin{bmatrix} n_x & t_x & b_x & p_x \\ n_y & t_y & b_y & p_y \\ n_z & t_z & b_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$

and substituting into the left-hand side of $(A_1^0)^{-1} T (A_6^5)^{-1} = A_2^1 A_3^2 A_4^3 A_5^4$

we obtain another expression of x_W^1

$$x_W^1 = \begin{bmatrix} p_x^* c_1 + p_y^* s_1 \\ -p_z^* + l_0 \\ -p_x^* s_1 + p_y^* c_1 \end{bmatrix}$$

where p_x^*, p_y^*, p_z^* represent the coordinates of point W, and are given by

$$p_x^* = p_x - l_2 b_x$$
$$p_y^* = p_y - l_2 b_y$$
$$p_z^* = p_z - l_2 b_z$$

Equating yields three equations with three unknowns:

$$d_3 s_2 = p_x^* c_1 + p_y^* s_1$$

$$-d_3 c_2 = -p_z^* + l_0$$

$$l_2 = -p_x^* s_1 + p_y^* c_1$$

To solve the last equation, we let: $t = \tan\left(\frac{\theta_1}{2}\right)$

so that

$$c_1 = \cos\theta_1 = \frac{1-t^2}{1+t^2} \quad \text{and} \quad s_1 = \sin\theta_1 = \frac{2t}{1+t^2}$$

Substituting expressions, we obtain a quadratic equation in terms of the unknown variable t :

$$(l_1 + p_y^*)t^2 + 2p_x^* t + l_1 - p_y^* = 0$$

Solving the above equation for t yields

$$\theta_1 = 2\tan^{-1}\left[\frac{-p_x^* \pm \sqrt{p_x^{*2} + p_y^{*2} - l_1^2}}{l_1 + p_y^*}\right]$$

Note that the quantity under the square root must be positive. Otherwise, the solution does not exist, which means that the specified end-effector position is out of the reachable range, or ***workspace***, of the manipulator arm.

Note also that the arctangent function can take two values, which are 180 degrees apart.

DIFFERENTIAL MOTION

in order to coordinate joint motions, we derive the *differential* relationship between the joint displacements and the end-effector location, and then solve for the individual joint motions.

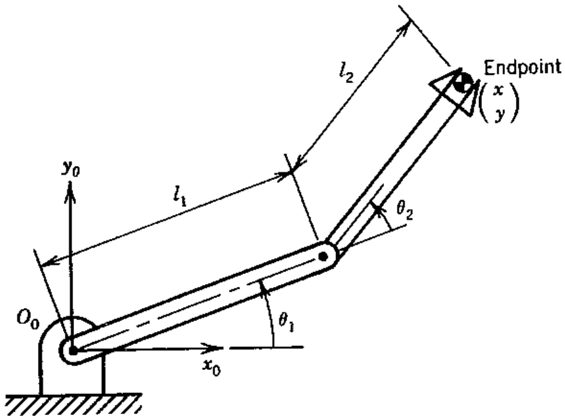


Figure 3-1 : Two degree-of-freedom planar manipulator.

$$x(\theta_1,\theta_2) = l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2)$$

$$y(\theta_1,\theta_2) = l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2)$$

$$dx = \frac{\partial x(\theta_1,\theta_2)}{\partial\theta_1}d\theta_1 + \frac{\partial x(\theta_1,\theta_2)}{\partial\theta_2}d\theta_2$$

$$dy = \frac{\partial y(\theta_1,\theta_2)}{\partial\theta_1}d\theta_1 + \frac{\partial y(\theta_1,\theta_2)}{\partial\theta_2}d\theta_2$$

In vector form the above can be written as

$$d\mathbf{x} = \mathbf{J}d\boldsymbol{\theta}$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial\theta_1} & \frac{\partial x}{\partial\theta_2} \\ \frac{\partial y}{\partial\theta_1} & \frac{\partial y}{\partial\theta_2} \end{bmatrix} = \begin{bmatrix} -l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) & -l_2\sin(\theta_1+\theta_2) \\ l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2) \end{bmatrix}$$ *Jacobian*

dividing both sides of by the infinitesimal time increment dt, we obtain the end-effector velocity vector.

$$\frac{d\mathbf{x}}{dt} = \mathbf{J}\frac{d\boldsymbol{\theta}}{dt}$$

that is,

$$\mathbf{v} = \mathbf{J}\dot{\boldsymbol{\theta}}$$

Singularity occurs when the determinant of the manipulator Jacobian is zero.

6 D.O.F

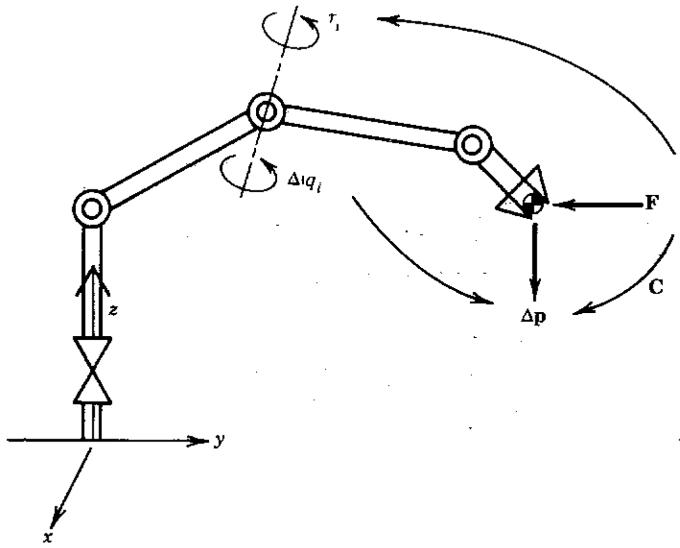
n = 2 PLANER
n = 3 3-D

The dimension of the Jacobian matrix is now $6 \times n$; the first three row vectors are associated with the linear velocity $\mathbf{v}_e$, while the last three correspond to the angular velocity ω_e. Each column vector, on the other hand, represents the velocity and angular velocity generated by the corresponding individual joint. Let us determine each column vector of the Jacobian matrix as functions of link parameters and arm configuration. Let $\vec{\mathbf{J}}_{Li}$ and $\vec{\mathbf{J}}_{Ai}$ be 3×1 column vectors of the Jacobian matrix associated with the linear and angular velocities, respectively. Namely, we partition the Jacobian matrix so that

$$\mathbf{J} = \begin{bmatrix} \vec{\mathbf{J}}_{L1} & \vec{\mathbf{J}}_{L2} & \cdots & \vec{\mathbf{J}}_{Ln} \\ \vec{\mathbf{J}}_{A1} & \vec{\mathbf{J}}_{A2} & \cdots & \vec{\mathbf{J}}_{An} \end{bmatrix}$$

STATICS

Endpoint Compliance Analysis



Endpoint compliance and joint servo stiffness.

$$\vec{\tau} = \mathbf{J}^T \vec{\mathbf{F}} \qquad \mathbf{J} = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) & -l_2 \sin(\theta_1+\theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix}$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1+\theta_2) & l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) \\ -l_2 \sin(\theta_1+\theta_2) & l_2 \cos(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

We model the stiffness of the drive system combined with the stiffness of the reducer and transmissions by a spring constant k_i that relates the deflection at joint i to the force or torque transmitted. Namely,

$$\vec{\tau}_i = k_i \Delta \vec{q}_i \qquad \mathbf{K} = \begin{bmatrix} k_1 & & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & & k_n \end{bmatrix}$$

where τ_i is the joint torque and Δq_i is the deflection at the joint axis.

the individual joint deflections $\Delta\mathbf{q}$ produce the endpoint deflection $\Delta\mathbf{p}$ according to

$$\Delta\mathbf{p} = \mathbf{J}\Delta\mathbf{q}$$

Substituting, we obtain

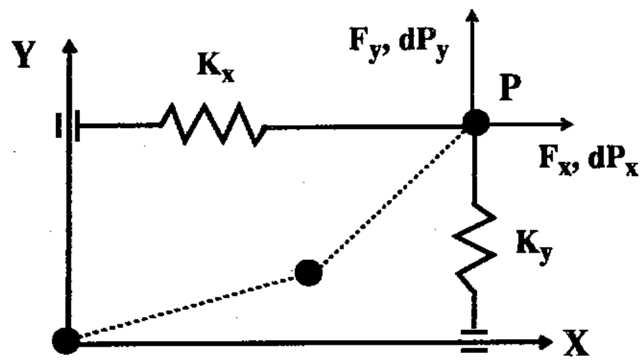
$$\Delta\vec{\mathbf{p}} = \mathbf{C}\vec{\mathbf{F}}$$

where

$$\mathbf{C} = \mathbf{J}\mathbf{K}^{-1}\mathbf{J}^T$$

Thus the deflection at the endpoint $\Delta\mathbf{p}$ is related to the endpoint force **F** by the $m \times m$ matrix **C**. The matrix **C** is called the *compliance matrix* of the arm endpoint.

The Cartesian Spring



$$\vec{\mathbf{F}} = \mathbf{K}\,\mathbf{dP}$$

$$\begin{bmatrix} F_x \\ F_y \end{bmatrix} = \begin{bmatrix} K_x & 0 \\ 0 & K_y \end{bmatrix} \begin{bmatrix} dP_x \\ dP_y \end{bmatrix}$$

$$\vec{\mathbf{T}} = (\mathbf{J}^T\,\mathbf{K}\,\mathbf{J})\,\mathbf{d\Theta}$$